DISCRETE MATH - CS275 FINAL EXAM - SOLUTION

DECEMBER 14TH 2004, 1PM - 3PM

Each of the 4 exercises is worth 5 points.
No calculator is needed or allowed in this exam.
Justify your answers.

Advice:

- Read all the exercises before answering to any.
- Answering the wrong question does not help. Read each exercise with great care and **without hurrying**. If needed, read it **many times**, until the meaning of the questions is clear.
- Check for extra information on the blackboard.

Exercise 1. For each of the pairs of simple graphs in Figure 1.1, prove that the two graphs are isomorphic by displaying a graph isomorphism, or explain why they are not isomorphic.

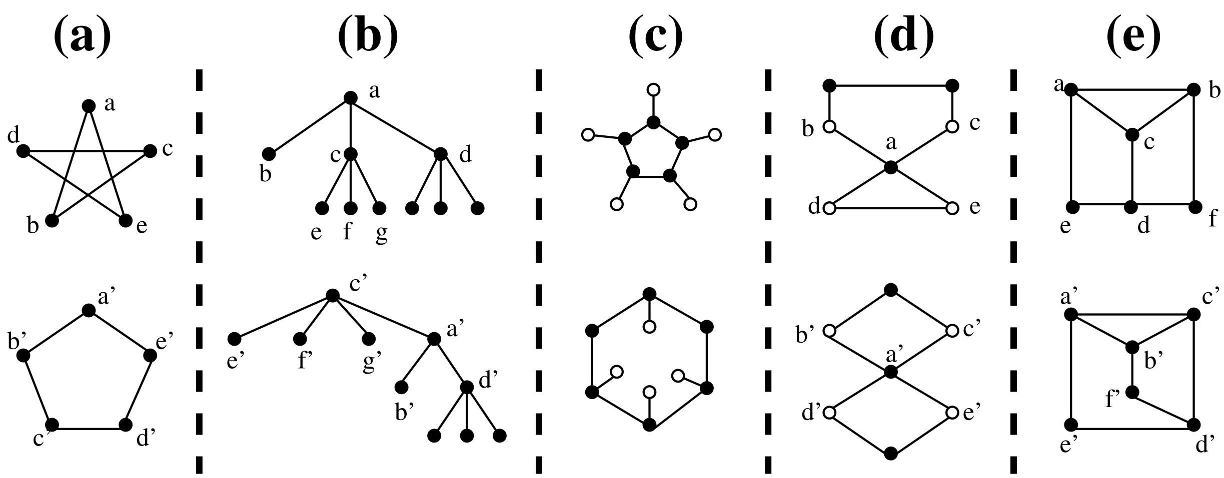

FIGURE 1.1. Graphs for Exercise 1.

a) Isomorphic: Consider e.g. the labeling in Fig. 1.1 (a) and the mapping $a \longrightarrow a'$, ..., $e \longrightarrow e'$.

b) Isomorphic: Consider e.g. the labeling in Fig. 1.1 (b) and the mapping $a \longrightarrow a'$, ...

c) Non-isomorphic: The topmost graph has five pendant vertices, while the bottom graph has just three.

d) Non-isomorphic: Proof by contradiction: assume there exists a graph isomorphism between these two graphs. Consider the labeling in Fig. 1.1. If the two graphs were isomorphic, then a, the only vertex of degree four would necessarily be mapped to a' and its neighbors $\{b, c, d, e\}$ would be mapped to $\{b', c', d', e'\}$. Now, in the topmost graph, two of these neighbors are adjacent, while this is not the case in the bottom graph.

e) Isomorphic: Consider e.g. the labeling in Fig. 1.1 (e) and the mapping $a \longrightarrow a'$, ..., $f \longrightarrow f'$.

Exercise 2. A six-sided dice is rolled 10 times. The outcome is represented by a sequence $X = (x_1, \ldots, x_{10})$, where $x_i \in \{1, \ldots, 6\}$ is the result of the i^{th} roll.

a) How many different outcomes are possible? Justify your answer.
Solution: The set of possible outcomes is $\{1, \ldots, 6\} \times \ldots \times \{1, \ldots, 6\}$, which has cardinal: 6^{10}.

b) What are the least and greatest possible values of $\sum_{i=1}^{10} x_i$? Justify your answer.
Solution: Since each x_i is not less than 1, their sum is not less than 10. Since each x_i is not more than 6, their sum is not more than $6 \cdot 10 = 60$.

c) Write in mathematical notation the set of outcomes in which exactly one six comes out.
Solution:

$$\begin{aligned} A &= \left\{X \in \{1, \ldots, 6\}^{10} \mid \exists! i \in \{1, \ldots, 10\}, \, x_i = 6\right\} \\ &= \{6\} \times \{1, \ldots, 5\}^9 \cup \\ &\quad \{1, \ldots, 5\} \times \{6\} \times \{1, \ldots, 5\}^8 \cup \\ &\quad \vdots \\ &\quad \{1, \ldots, 5\}^9 \times \{6\} \end{aligned}$$

d) How many different outcomes are there in which exactly one six comes out? Justify your answer.
Solution: An element in A above is uniquely determined by one index $i \in \{1, \ldots, 10\}$ and by the values, in $\{1, \ldots, 5\}^9$ of the nine other rolls. By the "product rule," the number of outcomes is $10 \cdot 5^9 = C(10, 1) \cdot 5^9$.

e) How many different outcomes are there in which $\sum_{i=1}^{10} x_i = 11$? Justify your answer.
Solution: A sum of 11 can only be obtained by adding nine ones and a two, i.e. $X \in \{(2, 1, \ldots, 1), (1, 2, 1 \ldots, 1), \ldots, (1, \ldots, 1, 2)\}$ and there are 10 possibilities.

Exercise 3. Let f and g be functions with domain $\mathbb{N}$ and co-domain $\mathbb{N}$, defined by:

$$\begin{aligned} f(x) &= 3\left\lceil \frac{x}{3} \right\rceil \quad \text{and} \\ g(x) &= x - 3\left\lceil \frac{x}{3} \right\rceil. \end{aligned}$$

Consider the following statements, which may be true or false:

A: $\forall y \in \mathbb{N}, \exists x \in \mathbb{N}, f(x) = y$.
B: $\forall y \in \mathbb{N}, \exists! x \in \mathbb{Z}, g(f(x)) = y$.
C: $\exists y \in \mathbb{N}, \forall x \in \mathbb{Z}, g(x) + f(x) \neq y$.

Answer the questions:

a) Write in English the statements **A**, **B** and **C**. If you like, you may use the words "one-to-one", "onto", "bijection" etc.
Solution:
A: "f is onto." or "all natural number y, has a pre-image by f," or ...
B: "The composition of f and g is bijective," or "for all natural number y, there is a unique natural number x such that $(g \circ f)(x)$ is equal to y," or ...
C: "The sum of g and f is not onto," or "there is a natural number y that has no pre-image by $f + g$," or ...

b) Prove or disprove each of the statements **A**, **B** and **C**.
Solution:
A: False: Can be proved by displaying a counter-example: since, for all $x \in \mathbb{N}$, $f(x)$ is a multiple of 3, 1 has no pre-image by f.
Common error: Not know that $0 \in \mathbb{N}$.
B: False: Can be proved by displaying a counter-example: Since $f(0) = 0$, $f(3) = 3$, $g(0) = 0$ and $g(3) = 0$, one has $g(f(0)) = 0 = g(f(3))$.
Can also be proven by noting that $(g \circ f)(x) = 0$ for all x. Indeed, since $f(x)$ is always a multiple of 3 and g maps multiples of 3 to 0, $g \circ f = 0$.
Can also be also proven by saying that since f is not one-to-one, $g \circ f$ cannot be one-to-one.
C: False: since $f(x) + g(x) = x$, the function $f + g$ is onto.

Hint: You may find useful to compute by hand a few values of $f(x)$ and $g(x)$.

Note: It should have been "$\mathbb{Z}$" all over and not "$\mathbb{N}$" in the wording of this exercise; no points were removed because of that.

Exercise 4. We consider strings (sequences) of characters "A", "B" and "C", in which consecutive consonants are not allowed; that is, substrings "BB", "BC", "CB" and "CC" are not allowed.
Let x_n be the number of allowed strings of length n that end in "B" or "C" and let y_n be the number of allowed strings of length n that end in "A".

a) Write all the allowed strings of length one, two and three and write x_1, x_2, x_3, y_1, y_2 and y_3.
Solution:
1) A, B, C.
2) AA, AB, AC, BA, CA.
3) AAA, AAB, AAC, ABA, ACA, BAA, BAB, BAC, CAA, CAB, CAC.

$x_1 = 2$, $x_2 = 2$, $x_3 = 6$, $y_1 = 1$, $y_2 = 3$ and $y_3 = 5$.

b) Notice that, for $n > 1$, all strings of length n ending in a consonant are obtained by appending a consonant to a message of length $n-1$ that ends in "A".
Write x_n as a function of y_{n-1}.
Solution: $x_n = 2y_{n-1}$.

c) Notice that, for $n > 1$, all strings of length n ending in "A" are obtained by appending an "A" to a message of length $n-1$.
Write y_n as a function of y_{n-1} and x_{n-1}.
Solution: $y_n = y_{n-1} + x_{n-1}$.

d) Using your answers to the previous questions, find a homogeneous recurrence relation verified by y_n. Solve this relation and write y_n as a function of n.
Solution: $y_n = y_{n-1} + 2y_{n-2}$.
The characteristic polynomial $r^2 - r - 2 = (r-2)(r+1) = 0$ has distinct roots $r_1 = 2$ and $r_2 = -1$, so that y_n has the form $y_n = \alpha_1 2^n + \alpha_2 (-1)^n$.
Solving

$$\begin{cases} y_1 &= 1 &= 2\alpha_1 - \alpha_2 \\ y_2 &= 3 &= 4\alpha_1 + \alpha_2, \end{cases}$$

one gets $\alpha_1 = \frac{2}{3}$ and $\alpha_2 = \frac{1}{3}$ and

$$y_n = \frac{1}{3}\left(2^{n+1} + (-1)^n\right).$$

It is worth checking that indeed $y_3 = \frac{1}{3}(16-1) = 5$ as was found above.

e) Write the total number of allowed strings, $x_n + y_n$ as a function of n only.
Solution: One may recall that $x_n + y_n = y_{n+1} = \frac{1}{3}\left(2^{n+2} - (-1)^n\right)$ or use the relation $x_n = 2y_{n-1}$ to get

$$\begin{aligned} x_n + y_n &= \frac{1}{3}\left(2^{n+1} + (-1)^n\right) + \frac{2}{3}\left(2^n + (-1)^{n-1}\right) \\ &= \frac{1}{3}\left(2^{n+1} + (-1)^n\right) + \frac{1}{3}\left(2^{n+1} - 2(-1)^n\right) \\ &= \frac{1}{3}\left(2^{n+2} - (-1)^n\right) \\ &= \frac{1}{3}\left(2^{n+2} + (-1)^{n+1}\right) \\ &= y_{n+1}. \end{aligned}$$