CS275 GRADED HOMEWORK 3

GIVE BACK ON TUESDAY SEP. 28TH 2004 AT BEGINNING OF CLASS

For each question, read **each word** with the **greatest care** and **without precipitation**. If you have doubts about what is asked, **go back** to the wording of the question until the meaning of the question is clear. Then try to find an answer. If you get stuck, don't hesitate to **contact** your T.A. or me.

Please write on your homework your section number and an estimate of the time you spent solving it.

Exercise 1.
Consider the rectangular city maps in Figure 1.1.

a) How many distinct ways are there to go from point A to point B, if one is only allowed to move along the lines, either Southward or Eastward.
Solution: All acceptable paths from A to B are 14 step long, where each step is either one block to the South or one block to the East. Of these 14 steps, exactly 5 are towards the East. It does not matter when these 5 steps are done amongst the 14 steps.
Each path can be represented as a sequence $X = (s_1, ..., s_{14})$, where s_i is "E" if the i^{th} step is Eastwards and "S" if it is southwards. Of the 14 steps, exactly 5 are eastwards, so that exactly 5 of the s_i are "S".
The number of ways of placing exactly 5 "E"s amongst 14 positions is

$$C\left(14, 5\right) = 2002.$$

b) How many distinct ways are there to go from point C to point D, if one is only allowed to move along the lines, either Southward or Eastward.
Solution: Any path must either 1) pass through C" and C' or 2) pass through D' and D". First case: there is 1 way of going from C to C' and $C\left(10, 5\right)$ ways of going from C" to D. Second case: there are $C\left(8, 3\right)$ ways of going from C to D' and 1 way of going from D" to D. In all there are thus

$$1 \cdot C\left(10, 5\right) + C\left(8, 3\right) \cdot 1 = 252 + 56 = 308$$

ways of going from C to D.

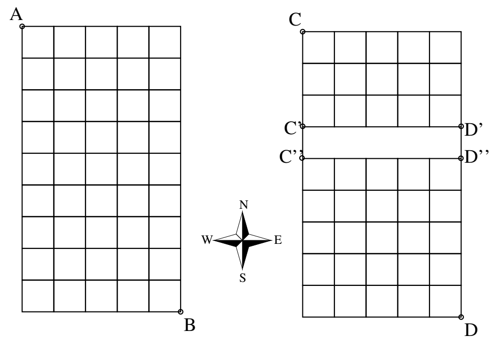

FIGURE 1.1. **Left:** Rectangular city plan. **Right:** Two-part city with two bridges.

Exercise 2. A lottery has 64 tickets. How many ways are there of giving 60 of these tickets to 60 persons, knowing that each person receives exactly one ticket? Justify your answer.
Solution: The 1st person is given one out of the 64 tickets.
The 2nd person gets one out of the 63 leftover tickets.
The 3rd person gets one out of the 62 leftover tickets.
...
The 60th person gets one out of the (64-59) leftover tickets.
Altogether, there are thus

$$64 \cdot 63 \cdot \ldots \cdot (64 - 60 + 1) = \frac{64!}{(64-60)!} = P(64, 60) \simeq 5.29E87.$$

Exercise 3. A web server answers 1000 queries per day. For each, a single line of text is added to a log file. At 1am, the server is put offline for a minute. During this time, half[1] the lines of the log file are deleted. Let x_m be the number of lines in the log file at 1:01am of the m^{th} day of operation of the server.

a) Taking into account the 1000 daily queries and the deletion process, write x_{m+1} as a function of x_m.
Solution: Number of lines at 1:01am of day m: x_m.
Number of lines at 1:00am of day $m+1$: $x_m + 1000$, since there were 1000 queries.
Number of lines at 1:01am of day $m+1$: $\frac{x_m+1000}{2}$, since half the lines have been removed.
Thus:

$$x_{m+1} = \frac{x_m + 1000}{2}.$$

b) On the first day, at 1:01am, the log file has 1000 lines, i.e. $x_1 = 1000$.
1) Compute x_2, x_3 and x_4.
Solution: $x_2 = \frac{1000+1000}{2} = 1000$. $x_3 = x_4 = 1000$.
2) What are x_{1000}, $x_{1000000}$ and $x_{1000000000}$?
Solution: $x_{1000} = x_{1000000} = x_{1000000000} = 1000$.

[1] Assume it is possible to keep a fractional number of lines.

3) Justify your answer to the last question with a proof by induction.
Solution:
Let $P(n)$ be the property $x_n = 1000$.
Basis step: $P(1)$ is true.
Induction step: Suppose $P(n)$ is true, i.e. $x_n = 1000$. Then $x_{n+1} = \frac{x_n+1000}{2} = 1000$, so that $P(n+1)$ is deduced from $P(n)$.

Exercise 4. For all $n \in \mathbb{N}$, define $x_n = 1 + \frac{1}{2} + \ldots + \frac{1}{2^n} = \sum_{i=0}^{n} \frac{1}{2^i}$.

a) Compute x_0, x_1, x_2, x_3 and x_4.

Solution: $x_0 = 1$, $x_1 = 1+\frac{1}{2} = \frac{3}{2} = 2-\frac{1}{2}$, $x_2 = \frac{7}{4} = 2-\frac{1}{4}$, $x_3 = \frac{15}{8} = 2-\frac{1}{8}$, $x_4 = \frac{31}{16} = 2 - \frac{1}{16}$.

b) Show by induction that, for all $n \in \mathbb{N}$, $x_n = 2 - \frac{1}{2^n}$.

Solution: Define $P(n) \stackrel{\Delta}{\equiv} x_n = 2 - \frac{1}{2^n}$.
Basis step: $x_0 = 1 = 2 - \frac{1}{2^0}$, so that $P(0)$ is true.
Induction step: Assume that $P(k)$ is true for some $k \in \mathbb{N}$. Then, $x_k = 2 - \frac{1}{2^k}$. As a consequence,

$$\begin{aligned} x_k + \frac{1}{2^{k+1}} &= 2 - \frac{1}{2^k} + \frac{1}{2^{k+1}} \\ &= 2 - \frac{2}{2 \cdot 2^k} + \frac{1}{2^{k+1}} \\ &= 2 - \frac{2}{2^{k+1}} + \frac{1}{2^{k+1}} \\ &= 2 + \frac{1-2}{2^{k+1}} \\ &= 2 - \frac{1}{2^{k+1}}. \end{aligned}$$

Since moreover, by definition of x_{k+1}, $x_{k+1} = x_k + \frac{1}{2^{k+1}}$ and thus $P(k+1)$ is true.

Exercise 5. The passwords on a computer system consist of characters. Each character belongs to the set $\mathcal{C} = \{$`'a'...'z'`, `'A'...'Z'`, `'0'...'9'`$\}$.

a) Write in mathematical notation the set of 4-character passwords. How many elements does it have?
Solution: The set of 4-character passwords is

$$\mathcal{C} \times \mathcal{C} \times \mathcal{C} \times \mathcal{C} = \mathcal{C}^4 = \{(a_1, a_2, a_3, a_4) \mid \forall i,\ a_i \in \mathcal{C}\}$$

Its cardinal is: $\left|\mathcal{C}^4\right| = |\mathcal{C}|^4 = 62^4 = 14,776,336$.

b) Write in mathematical notation the set of 4-, 5- or 6- character passwords? What is its cardinal?
Solution: The set of 4-, 5- or 6-character passwords is

$$\mathcal{C}^4 \cup \mathcal{C}^5 \cup \mathcal{C}^6 = \left\{P \mid P \in \mathcal{C}^4 \vee P \in \mathcal{C}^5 \vee P \in \mathcal{C}^6\right\}.$$

Its cardinal is

$$\left|\mathcal{C}^4 \cup \mathcal{C}^5 \cup \mathcal{C}^6\right| = \left|\mathcal{C}^4\right| + \left|\mathcal{C}^5\right| + \left|\mathcal{C}^6\right| = |\mathcal{C}|^4 + |\mathcal{C}|^5 + |\mathcal{C}|^6 \simeq 5.77E10.$$

c) Write in mathematical notation the set of 4-character passwords consisting only of upper- or lower-case letters? How many such passwords are there?
Solution: Let $\mathcal{C}' = \{$`'a'...'z'`, `'A'...'Z'`$\}$. The set of 4-character passwords consisting of letters is

$$\mathcal{C}'^4 = \{(a_1, a_2, a_3, a_4) \mid \forall i,\ a_i \in \mathcal{C}'\}.$$

The cardinal of this set is

$$52^4 = 7,311,616.$$

d) Write in mathematical notation the set of 4-character passwords with 1 or more digits in them? How many such passwords are there?
Solution: Since this set is the complementary of the set of passwords with no digits in them, this set is

$$\mathcal{C}^4 \setminus \mathcal{C}'^4 = \{(a_1, a_2, a_3, a_4) \mid \forall i,\ a_i \in \mathcal{C},\ \exists i\, a_i \in \mathcal{C} \setminus \mathcal{C}'\}.$$

Since $\mathcal{C}' \subseteq \mathcal{C}$, one has

$$\left|\mathcal{C}^4 \backslash \mathcal{C}'^4\right| = \left|\mathcal{C}^4\right| - \left|\mathcal{C}'^4\right| = 62^4 - 52^4 = 7,464,720.$$

e) Write in mathematical notation the set of 4-character passwords with exactly 3 digits in them? How many such passwords are there?
Solution: Let $\mathcal{D} = \{0...9\}$. The set in question is

$$\begin{aligned}\left\{(c_1, c_2, c_3, c_4) \in \mathcal{C}^4 \mid \exists! i \in \{1...4\},\ c_i \in \mathcal{C}'\right\} &= \\ & \mathcal{D} \times \mathcal{D} \times \mathcal{D} \times \mathcal{C} \\ \cup\ & \mathcal{D} \times \mathcal{D} \times \mathcal{C} \times \mathcal{D} \\ \cup\ & \mathcal{D} \times \mathcal{C} \times \mathcal{D} \times \mathcal{D} \\ \cup\ & \mathcal{C} \times \mathcal{D} \times \mathcal{D} \times \mathcal{D}.\end{aligned}$$

Its cardinal is

$$4 \cdot 26^1 \cdot 10^3 = 104,000.$$

Exercise 6. Solve Exercise 8 p. 310 of the textbook [1]: how many different three-letter initials with none of the letters repeated can people have?
Solution: $P(26,3) = 26 \cdot 25 \cdot 24 = 15,600$

Exercise 7. Solve Exercise 26 p. 311 of the textbook [1]: how many license plates can be made using either three letters followed by three digits or four letters followed by two digits?
Solution: Summation rule: $26^3 \cdot 10^3 + 26^4 \cdot 10^2 = 17,576,000 + 45,697,600 = 63,273,600 \simeq 6.3E7$.

Exercise 8. Solve Exercise 20 p. 325 of the textbook [1]: how many bit strings of length ten have

a) exactly three 0s?
Solution: $C(10,3) = 120$
b) more 0s than 1s?
Solution: $C(10,6) + C(10,7) + C(10,8) + C(10,9) + C(10,10) = 386$
c) at least seven 1s?
Solution: $C(10,7) + C(10,8) + C(10,9) + C(10,10) = 176$
d) at least three 1s?
Solution: $2^2 - C(10,0) - C(10,1) - C(10,2) = 968$

Exercise 9. How many sequences $(a_1, a_2, a_3, a_4, a_5)$ of five numbers taken in $\{0, ..., 9\}$ are such that $a_1 < a_2 < a_3 < a_4 < a_5$? Justify your answer.

Solution: Any such sequence can be defined by a subset A of five elements in $\{0, ..., 9\}$. For any such A, one can sort in increasing order its elements and call them $a_1, ..., a_5$, thus defining the sequence $(a_1, ..., a_5)$. It is clear that two distinct sets define distinct sequences and that all sequences of five increasing elements in $\{0, ..., 9\}$ can be defined by a subset of five elements in $\{0, ..., 9\}$. There are thus as many sequences as 5-element subsets of the set $\{0, ..., 9\}$, so there are

$$C(10, 5) = 252$$

sequences.

More in details. The set we are interested in is

$$\mathcal{S} = \left\{(a_1, a_2, a_3, a_4, a_5) \in \{0..9\}^5 \mid a_1 < a_2 < a_3 < a_4 < a_5\right\}.$$

We will define a one-to-one and onto mapping from the set of 5-element subsets of $\{0, ..., 9\}$ into $\mathcal{S}$, and thus show that this last set has $C(10, 5)$ elements.

To any subset A of $\{0, ..., 9\}$ with five elements, we associate the sequence $S(A) = (a_1, a_2, a_3, a_4, a_5)$ of its sorted elements. We thus define a mapping from the 5-element subsets of $\{0, ..., 9\}$ into the set of sorted sequences of 5 digits.

This mapping is onto, since it allows to define any sequence $(a_1, ..., a_5)$, since $S(\{a_1, ..., a_5\}) = (a_1, ..., a_5)$. This mapping is one-to-one, since distinct sets yield distinct sequences.

The cardinal of $\mathcal{S}$ is thus equal to the cardinal of the set of 5-element subsets of $\{0, ..., 9\}$, which is $C(10, 5)$, as seen in class.

References

[1] K. H. Rosen. *Discrete Mathematics and Its Applications.* Mc Graw Hill, 5 edition, 2003.