CS275 GRADED HOMEWORK 4 - PROVISIONAL SOLUTION

GIVE BACK ON TUESDAY OCTOBER. 19TH 2004 AT BEGINNING OF CLASS

For each question, read **each word** with the greatest care and **without hurrying**. If you have doubts about what is asked, **go back** to the wording of the question until the meaning of the question is clear. Then try to find an answer. If you get stuck, don't hesitate to **contact** your T.A. or me.

Please write your section number on your homework as well as a rough estimate of the time you spent solving it.

Exercise 1. In town, there are just 3 restaurants:

- A seafood restaurant serving dishes from a "set" $\mathcal{S}$.
- A vegetarian restaurant serving dishes from a "set" $\mathcal{V}$. $\mathcal{S} \cap \mathcal{V} = \emptyset$.
- A carnivorous restaurant serving dishes from a "set" $\mathcal{C}$. $\mathcal{S} \cap \mathcal{C} = \mathcal{V} \cap \mathcal{C} = \emptyset$.

A dinner consists of one dish or two dishes (the order matters) from the same restaurant and ends with either a cup of coffee, a cup of liquor, a mint or nothing. Let $\mathcal{E} = \{\text{Coffee, Liquor, Mint, Nothing}\}$ be the set of possible endings.

a) Write in mathematical notation the set of possible dinners.
Solution: A dinner could consists of a sequence $(s, e) \in \mathcal{S} \times \mathcal{E}$ consisting of a seafood dish s and an ending e; or a sequence $(v, v', e) \in \mathcal{V} \times \mathcal{V} \times \mathcal{E}$ consisting of two vegetarian dishes and an ending; etc. Joining all the possibilities, the set of dinners can be written:

$$\begin{aligned}
& \big\{(d, e) \mid d \in \mathcal{S} \cup \mathcal{V} \cup \mathcal{C},\ e \in \mathcal{E}\big\} \cup \big\{(d, d', e) \mid (d,\ d') \in \mathcal{S}^2 \cup \mathcal{V}^2 \cup \mathcal{C}^2,\ e \in \mathcal{E}\big\} \\
= \ & \mathcal{S} \times \mathcal{E} \ \cup\ \mathcal{S} \times \mathcal{S} \times \mathcal{E} \ \cup\ \mathcal{V} \times \mathcal{E} \ \cup\ \mathcal{V} \times \mathcal{V} \times \mathcal{E} \ \cup\ \mathcal{C} \times \mathcal{E} \ \cup\ \mathcal{C} \times \mathcal{C} \times \mathcal{E} \\
= \ & (\mathcal{S} \cup \mathcal{V} \cup \mathcal{C}) \times \mathcal{E} \ \cup\ \left(\mathcal{S}^2 \cup \mathcal{V}^2 \cup \mathcal{C}^2\right) \times \mathcal{E}
\end{aligned}$$

b) What is its cardinal? I.e. in how many different ways can one have dinner?
Solution: Since the sets in the union above are two-by-two disjoint, the cardinal of this set, given by the product and sum rules, is:

$$|\mathcal{S}| \cdot |\mathcal{E}| + |\mathcal{S}|^2 \cdot |\mathcal{E}| + |\mathcal{V}| \cdot |\mathcal{E}| + |\mathcal{V}|^2 \cdot |\mathcal{E}| + |\mathcal{C}| \cdot |\mathcal{E}| + |\mathcal{C}|^2 \cdot |\mathcal{E}|.$$

c) What is the cardinal of the set of dinners in which two distinct dishes are served?
Solution: There are $P(|\mathcal{S}|, 2)$ sequences without repetition of two seafood dishes. Altogether, there are

$$(P(|\mathcal{S}|, 2) + P(|\mathcal{V}|, 2) + P(|\mathcal{C}|, 2)) \cdot |\mathcal{E}|$$

dinners in which two distinct dishes are served.

Exercise 2. Exercise 8 p. 109 of [1]: Find these values.

a) $\lfloor 1.1 \rfloor$ 1
b) $\lceil 1.1 \rceil$ 2
c) $\lfloor -0.1 \rfloor$ .... -1
d) $\lceil -0.1 \rceil$ 0
e) $\lceil 2.99 \rceil$ 3
f) $\lceil -2.99 \rceil$ .... -2
g) $\left\lfloor \frac{1}{2} + \left\lceil \frac{1}{2} \right\rceil \right\rfloor$ .... $\left\lfloor \frac{1}{2} + 1 \right\rfloor = 1$
h) $\left\lceil \left\lfloor \frac{1}{2} \right\rfloor + \left\lceil \frac{1}{2} \right\rceil + \frac{1}{2} \right\rceil$ .... $\left\lceil 0 + 1 + \frac{1}{2} \right\rceil = 2$

Exercise 3. For each of the functions below, determine whether it is onto and/or one-to-one.

a) $f : n \in \mathbb{N} \longrightarrow n+1 \in \mathbb{N}$
Solution: Not onto because 0 isn´t the image by f of a natural number: $\forall n \in \mathbb{N}, n+1 \neq 0$.
One-to-one because $m \neq n \Longrightarrow m+1 \neq n+1 \iff f(m) \neq f(n)$.

b) $f : x \in \mathbb{Z} \longrightarrow x^3 \in \mathbb{Z}$
Solution: Not onto because 2 isn´t the image by f of a natural number: $\forall n \in \mathbb{N}, n^3 \neq 2$.
One-to-one because $m \neq n \Longrightarrow m^3 \neq n^3 \iff f(m) \neq f(n)$.

c) $f : x \in \mathbb{R} \longrightarrow x^3 \in \mathbb{R}$
Solution: Onto because $\forall y \in \mathbb{R}$, $\exists x = \sqrt[3]{y} \in \mathbb{R}$ and $f(x) = y$.
One-to-one because $x \neq y \Longrightarrow x^3 \neq y^3 \iff f(x) \neq f(y)$.

d) $f : n \in \mathbb{Z} \longrightarrow \lceil n/2 \rceil \in \mathbb{Z}$
Solution: Onto because, $\forall n \in \mathbb{Z}$, $n = f(2n)$.
Not one-to-one because $f(2) = f(3) = 1$.

e) $f : n \in \mathbb{N} \longrightarrow (-1)^n \lfloor n/2 \rfloor \in \mathbb{Z}$
Solution: Note: $f(0) = 0$, $f(1) = 0$, $f(2) = 1$, $f(3) = -1$, $f(4) = 2$, $f(5) = -2$ ...
Onto because, if $n \geq 0$, $n = f(2n)$ and, if $n < 0$, $n = f(-2n+1)$.
Not one-to-one because $f(0) = f(1) = 0$.

Exercise 4. Let $A = \{1 \dots 10\}$ and $B = \{1 \dots 20\}$ and define the functions

$$\begin{array}{llcl} f: & x \in A & \longrightarrow & 2x \in B \\ g: & x \in B & \longrightarrow & \left\lceil \frac{x}{2} \right\rceil \in A \end{array}$$

a) Prove or disprove that f is onto and that it is one-to-one.
Solution: Not onto because, $1 \in B$ and $\forall m \in A$, $2m \neq 1$.
One-to-one because $m \neq n \Longrightarrow f(m) \neq f(n)$.

b) Prove or disprove that g is onto and that it is one-to-one.
Solution: Onto because, $\forall m \in A$, $2m \in B$ and $\forall g(2m) = m$.
Not one-to-one because $f(1) = f(2)$.

c) Prove or disprove that $g \circ f$ is onto and that it is one-to-one.
Solution: Note that, $\forall m \in A$, $(g \circ f)(m) = \left\lceil \frac{2m}{2} \right\rceil = \lceil m \rceil = m$.
This function is clearly onto and one-to-one.

Exercise 5. Let R be a relation on a set A containing two or more elements. Prove or disprove the following statements

a) If R is symmetric and transitive, then R is reflexive.
Solution: False: If $R = \emptyset$, i.e. no elements verify $R(x, y)$, then $(A, \emptyset)$ is symmetric and transitive (just check the definitions), but not reflexive.

b) If R is an equivalence relation, then R is not a total order relation.
Solution: True: Proof by contradiction.
Suppose R is an equivalence relation and a total order.
By assumption, there exist two distinct elements $x \neq y$ of A.
Since R is a total order, either $R(x, y)$ is true, or $R(y, x)$ is true. In the first case, since R is symmetric[1], one also has $R(y, x)$. In the second case, one also has $R(x, y)$. Thus, in all cases, both $R(x, y)$ and $R(y, x)$ are true.
Since R is antisymmetric[2], one has $x = y$, which contradicts the initial statement $x \neq y$.

[1] Because it is an equivalence relation.
[2] Because R is an order relation.

c) If R is an equivalence relation, then R is not a partial order relation.
Solution: False: Take $R = \{(x,x) \mid x \in A\}$, then (A,R) is an equivalence relation and also a poset (as seen in class).

d) If R is symmetric and antisymmetric, then $\forall x, y \in A$, $R(x,y) \implies x = y$.
Solution: True: If $R(x,y)$ is true, then, by symmetry, $R(y,x)$ is also true. Since then, $R(y,x)$ and $R(x,y)$ are true, one has, by antisymmetry, $x = y$.

Exercise 6. For each of the relations below

a) $A = \mathbb{Z}$, $R(x,y) \equiv x = \max\{x,y\}$
b) $A = \mathbb{Z}$, $R = \{(x,y) \mid x \leq y \wedge x^2 - y^2 = 0\}$
c) $A = \mathbb{Z}$, $R = \{(x,y) \mid x \leq y \vee x^2 - y^2 = 0\}$

1): Circle, on the grids[3] $\{-4,...,4\} \times \{-4,...,4\}$ of Figure 6.1, the points (x,y) that verify $R(x,y)$.
Solution: See Figure 6.1.

2): Prove or disprove that R is reflexive, symmetric, antisymmetric, transitive. Don´t forget that the R is defined on $\mathbb{Z}$, not just $\{-4,\ldots,4\}$.

Solution:

a) Note that $x = \max\{x,y\}$ iff $x \geq y$. R is reflexive, since $\forall x$, $x = \max\{x,x\}$. Not symmetric, since $R(2,1)$, but not $R(1,2)$. Antisymmetric, since $x \geq y$ and $y \geq x$ implies $x = y$. Transitive, since $x \geq y$ and $y \geq z$ implies $x \geq z$.
You could also just say that, since $(\mathbb{Z}, R)$ is the well-known poset $(\mathbb{Z}, \geq)$, R is reflexive, not symmetric, antisymmetric and transitive.
b) R is reflexive, since $\forall x$, $x \leq x \wedge x^2 - x^2 = 0$. Not symmetric, since $R(-1,1)$, but not $R(1,-1)$. Antisymmetric, since $x \geq y$ and $y \geq x$ implies $x = y$. Transitive, since $x \leq y \wedge y \leq z \implies x \leq z$ and $x^2 = y^2 \wedge y^2 = z^2 \implies x^2 = z^2$.
c) R is reflexive, since $\forall x$, $x \leq x \vee x^2 - x^2 = 0$. Not symmetric, since $R(0,1)$, but not $R(1,0)$. Not antisymmetric, since $R(-1,1)$ and $R(1,-1)$. Transitive, since $x \leq y \wedge y \leq z \implies x \leq z$ and $x^2 = y^2 \wedge y^2 = z^2 \implies x^2 = z^2$.

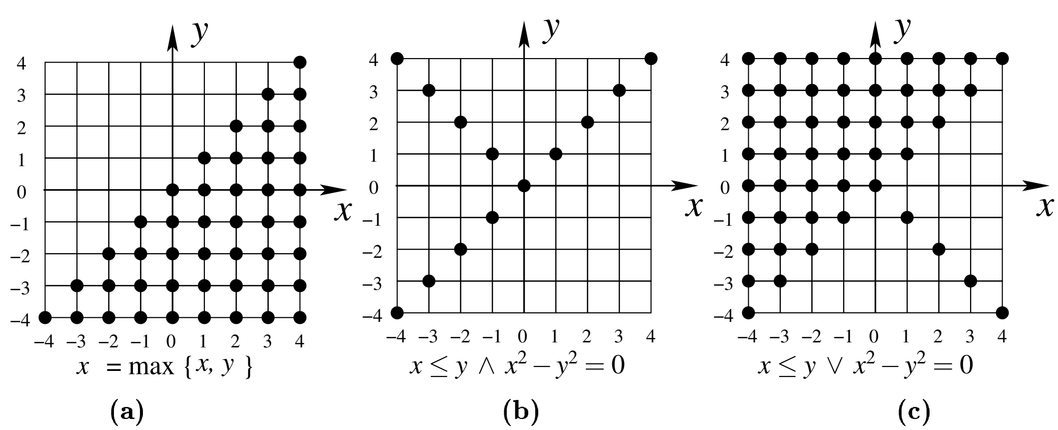

FIGURE 6.1. Grids for Exercise 6. Don´t forget that the relations are defined on $\mathbb{Z}$ and not just $\{-4\ldots4\}$. These grids are here only to illustrate "what the relations look like".

[3]You may copy these grids on any sheet of paper.

Exercise 7. Let $A = \{0,1\}^3 \cup \{0,1\}^4 \cup \{0,1\}^5 \ldots$ be the set of bit strings of length 3 or more and let R be the relation on A consisting of all pairs (x, y) such that x and y are equal except perhaps in their first three bits.

Solution: Notation: we will write $x = (x_1, x_2, x_3, \ldots, x_n)$, $y = (y_1, \ldots, y_m)$ elements of A. The relation R can then be written

$$R(x,y) \;\equiv\; m = n \wedge \forall i > 3,\, x_i = y_i.$$

a) Show that R is an equivalence relation on A.
Solution: This proof is trivial: R is reflexive because $\forall x \in A$, x is the same length as itself, and $\forall 3 < i < n$, $x_i = x_i$. Likewise, R is symmetric. And it is transitive: if $z = (z_1, ..., z_p)$ and $n = m$ and $\forall 3 < i \leq n$, $x_i = y_i$ and $m = p$ and $\forall 3 < i \leq m$, $y_i = z_i$, then $n = p$ and $\forall 3 < i \leq n$, $x_i = z_i$.

b) Write in mathematical notation the equivalence classes of the strings

1) 110
Solution:

$$\begin{aligned} \overline{110} &= \{y \in A \mid R(110, y)\} \\ &= \{(y_1, y_2, y_3) \mid \forall 1 \leq i \leq 3,\, y_i \in \{0,1\}\} \\ &= \{0,1\}^3 \end{aligned}$$

2) 1010
Solution:

$$\begin{aligned} \overline{1010} &= \{y \in A \mid R(1010, y)\} \\ &= \{(y_1, y_2, y_3, 0) \mid \forall 1 \leq i \leq 3,\, y_i \in \{0,1\}\} \\ &= \{0,1\}^3 \times \{0\} \end{aligned}$$

3) 11110
Solution:

$$\begin{aligned} \overline{11110} &= \{y \in A \mid R(11110, y)\} \\ &= \{(y_1, y_2, y_3, 1, 0) \mid \forall 1 \leq i \leq 3,\, y_i \in \{0,1\}\} \\ &= \{0,1\}^3 \times \{1\} \times \{0\} \end{aligned}$$

c) Write in lexicographic order 4 elements of each of the equivalence classes above.
Solution: Writing $\preceq$ the lexicographic order, one has e.g.

1) $000 \preceq 001 \preceq 010 \preceq 011$
2) $1000 \preceq 1010 \preceq 1100 \preceq 1110$
3) $00010 \preceq 10010 \preceq 11010 \preceq 11110$

Note: This exercise is largely inspired by Exercises 8 p. 513 and 25 p. 514 of [1].

References

[1] K. H. Rosen. *Discrete Mathematics and Its Applications.* Mc Graw Hill, 5 edition, 2003.