# CS275 GRADED HOMEWORK 5

GIVE BACK ON TUESDAY OCTOBER. 26TH 2004 AT BEGINNING OF CLASS

For each question, read **each word** with the greatest care and **without hurrying**. If you have doubts about what is asked, **go back** to the wording of the question until the meaning of the question is clear. Then try to find an answer. If you get stuck, don't hesitate to **contact** your T.A. or me.

Please write your section number on your homework as well as a rough estimate of the time you spent solving it.

## 1. Reminder about the class of Tuesday Oct. 19th.

**Proposition 1.** *Generalized pigeonhole principle. Let $x_1, \dots, x_k$ be real numbers.*
*If $\sum_{i=1}^{k} x_i = n$ for some number $n \in \mathbb{R}$, then, $\exists i \in \{1, \dots, k\}\,,\, x_i \geq \frac{n}{k}$.*

If furthermore the $x_i$ are all integers, then $\exists i \in \{1, \dots, k\}\,,\, x_i \geq \left\lceil \frac{n}{k} \right\rceil$. This is Theorem 2 p. 314 of [1].
The following was also shown in class.

**Proposition 2.** *Let $x_1, \dots, x_k \in \mathbb{R}$. If $\sum_{i=1}^{k} x_i = n$ for some $n \in \mathbb{R}$, then, $\exists i \in \{1, \dots, k\}\,,\, x_i \leq \frac{n}{k}$.*

This property was also shown.

**Proposition 3.** *Let $x_1, \dots, x_k$ be natural numbers.*
*If these numbers are two-by-two distinct, then $\max\{x_i \mid 1 \leq i \leq k\} \geq k - 1$ and $\sum_{i=1}^{k} x_i \geq \frac{(k-1)k}{2}$.*

The converse of this proposition is also of interest:

**Proposition 4.** *Let $x_1, \dots, x_k$ be natural numbers.*
*If either $\max\{x_i \mid 1 \leq i \leq k\} < k - 1$ or $\sum_{i=1}^{k} x_i < \frac{(k-1)k}{2}$, then $x_i = x_j$ for some $i \neq j$.*

Also, the definition of a subsequence of a sequence was given. The definition below corresponds to that on p. 317 of [1].

**Definition.** Let $X = (x_1, \dots, x_N)$ be a sequence of length $N$ and $1 \leq i_1 < \ldots < i_M \leq N$ be a strictly increasing sequence of integers. Then the sequence

$$x_{i_1},\, x_{i_2}, \dots, x_{i_M}$$

is called the **subsequence** of $X$ defined by $(i_1, \dots, i_M)$.

The four propositions and the definition above may or may not be useful in solving the following exercises.

## 2. Homework exercises

Please **justify each of your answers**.

**Exercise 1.** Let $R$ be the relation on natural numbers $R(x, y) \equiv y \in \{x,\, x + 2\}$. You may find helpful to draw a figure representing this relation.

**a)** Determine the basic properties of $R$ : is it reflexive, symmetric, antisymmetric, transitive?
**Solution:**
Reflexive: $x \in \{x, x + 2\}$;
Not symmetric: $R(0, 2)$, but not $R(2, 0)$.
Antisymmetric: By contradiction: if $x \in \{y, y + 2\} \wedge y \in \{x, x + 2\}$ and $x \neq y$, then $x = y + 2$ and thus $y \notin \{x, x + 2\} = \{y + 2, y + 5\}$.
Not transitive $R(0, 2)$, $R(2, 4)$, but not $R(0, 4)$.

**b)** Find all the elements $y$ such that $R^2(0,y)$ is true, and all the $y$ s.t. $R^2(1,y)$ is true.
**Solution:**

$$\begin{aligned}
\{y \mid R^2(0,y)\} &= \{y \mid \exists x\, R(0,x) \wedge R(x,y)\} = \{y \mid \exists x \in \{0,2\} \wedge R(x,y)\}\\
&= \{y \mid y \in \{0,2\} \vee y \in \{2,4\}\} = \{0,2,4\}.\\
\{y \mid R^2(1,y)\} &= \{y \mid \exists x\, R(1,x) \wedge R(x,y)\} = \{y \mid \exists x \in \{1,3\} \wedge R(x,y)\}\\
&= \{y \mid y \in \{1,3\} \vee y \in \{1,3\}\} = \{1,3,5\}.
\end{aligned}$$

**c)** Define in mathematical notation the relations $R^2$ and $R^n$, for any $n \in \mathbb{N}$.
**Solution:**

$$\begin{aligned}
R^2(x,y) &\equiv \exists x \in \mathbb{N}\, R(x,z) \wedge R(z,y)\\
R^n(x,y) &\equiv \exists x_1,\ldots,x_{n-1} \in \mathbb{N}\, R(x,x_1) \wedge R(x_{n-1},y) \wedge \forall i \in \{1\ldots n-2\},\, R(x_i,x_{i+1}).
\end{aligned}$$

**d)** Is the relation $R^*$ an order relation on the natural numbers?
**Solution:** Yes: it is easy to show that $R^*(x,y)$ is true iff $x-y$ is an even non-negative integer. It is easy to show that $R^*$ is reflexive and antisymmetric and we saw in class that $R^*$ is the transitive closure of $R$ and it is transitive.

**e)** Is the relation $R^*$ a total order on the natural numbers?
**Solution:** No: Neither $R^*(0,1)$ nor $R^*(1,0)$ are true and thus 0 and 1 are not comparable.

**Exercise 2.** Let $f$ be a function from a set $A$ to a set $B$ and let $S$ and $T$ be subsets of $A$.

**a)** Show that $f(S \cup T) = f(S) \cup f(T)$.
**Solution:**

$$\begin{aligned}
F(S \cup T) &= \{y \in B \mid \exists y \in S \cup T,\ f(x) = y\}\\
&= \{y \in B \mid \exists y \in S,\ f(x) = y \vee \exists y \in T,\ f(x) = y\}\\
&= \{y \in B \mid \exists y \in S,\ f(x) = y\} \cup \{y \in B \mid \exists y \in T,\ f(x) = y\}\\
&= f(S) \cup f(T)
\end{aligned}$$

**b)** Show that $f(S \cap T) \subseteq f(S) \cap f(T)$.
**Solution:**

$$\begin{aligned}
y \in F(S \cap T) &\iff \exists x \in S \cap T,\ f(x) = y\\
&\iff \exists x, x \in S \wedge x \in T \wedge f(x) = y\\
&\implies (\exists x, x \in S \wedge f(x) = y) \wedge (\exists x,\ x \in T \wedge f(x) = y)\\
&\iff f(S) \cap f(T)
\end{aligned}$$

**c)** Show that $S \subseteq T \implies f(S \cap T) = f(S) \cap f(T)$.
**Solution:** If $S \subseteq T$, then $f(S) \subseteq f(T)$ (easy) and $f(S) \cap f(T) = f(S)$. Since moreover $S \cap T = S$, one has $f(S \cap T) = f(S) = f(S) \cap f(T)$.

**d)** Find sets $A$, $B$, $S$, $T$ and a function $f$ such that $f(S \cap T) \subset f(S) \cap f(T)$.
**Solution:** Let $f : x \in \mathbb{N} \longrightarrow 1 \in \mathbb{N}$, and $S = \{0\}$ and $T = \{1\}$. Then $f(S \cap T) = f(\emptyset) = \emptyset \subset \{1\} = \{1\} \cap \{1\} = f(S) \cap f(T)$.

This is an augmented version of Exercise 32 p. 109 of [1].

**Exercise 3.** In a deck, there are 32 cards. Each card is of one of four possible suits and one of eight possible kinds, so that the deck can be identified with the set

$$\mathcal{D} = \{(s,k) \mid s \in \{1,2,3,4\},\ k \in \{1,\ldots,8\}\}.$$

**a)** How big should a set of cards be, to guarantee that two or more of its cards are of same kind?
**Solution:** 9, since there are 8 kinds of cards.

**b)** How many different sets of five cards do not have two or more cards of the same kind?
**Solution:** A five-hand consisting of cards of different kinds can be defined by choosing a set (combination) of 5 out of 8 kinds and 5 suits. The number of possibilities is

$$C(8,5)\cdot 4^5 = 57,344.$$

**c)** In a set of five cards with distinct kinds, can no three cards have consecutive kinds?
**Solution:** Yes: take e.g. kinds 1, 3, 5, 7, 8. There are no three consecutive kinds in this sequence.

**d)** In a set of five cards with distinct kinds, can no two cards have consecutive kinds?
**Solution:** No. Proof by contradiction. Take a set of 5 cards and order the cards so the kinds are non-decreasing: $(s_1,k_1),\ldots,(s_5,k_5)$ s.t. $k_1 \le k_2 \ldots \le k_5$. If there are no two cards with consecutive kinds and the kinds are all distinct, then $k_{i+1} \ge k_i + 2$ for all $1 \le i < 5$. Since $k_1 \ge 1$, one has $k_2 \ge 3$, $k_3 \ge 5$, $k_4 \ge 7$ and $k_5 \ge 9$, which is a contradiction with the assumption $k_i \in \{1\ldots 8\}$.

**Exercise 4.** A shop sells 5 types of donuts. One day, 14 clients came and bought a total of 96 donuts. Each client bought one or more donuts. Justify your answers to the following questions:

**a)** Can you show that one type of donuts at least was sold in 20 or more exemplars?
**Solution:** Yes: if each type of donut had been sold in 19 or less exemplars, at most $5\cdot 19 = 95$ donuts could have been sold. This can more formally be shown using the pigeonhole principle, as stated in Prop. 1 above.

**b)** Can you show that the least sold type of donut was sold in 19 or less exemplars?
**Solution:** Yes: if each type of donut had been sold in 20 or more exemplars, $5\cdot 20 = 100$ donuts at least would have been sold. Thus since one type of donut was sold in 19 or less exemplars, the least sold was also sold in 19 or less exemplar. This can more formally be shown Prop. 1 above.

**c)** Can you show that two of the clients bought the same number of donuts?
**Solution:** Yes: Let $x_i$, for $1 \le i \le 14$ be the number of donuts bought by client $i$. For all $i$, $x_i \ge 1$, since each client buys one donut at least. Since the sum of the 14 smallest integers is 105 and $x_1 + \ldots + x_{14} = 96 < 105$, two at least of the $x_i$ are equal.
**Note:** You **cannot** apply Prop. 4 here, since 96 is greater than $14\cdot 13/2 = 91$.

**Exercise 5.** Exercise 10 p. 708 of [1]: Show that $F(x,y,z) = xy + xz + yz$ has the value 1 if and only if at least two of the variables $x$, $y$ and $z$ have the value 1.
**Solution:** The rows of the truth table of $F$ has a 1 only on rows in which two variables at least are 1.

| $x$ | $y$ | $z$ | $F(x,y,z)$ |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 |
| 0 | 1 | 0 | 0 |
| 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | 1 |
| 1 | 1 | 0 | 1 |
| 1 | 1 | 1 | 1 |

**Exercise 6.** Exercise 4 p. 712 of [1]: Find the sum-of-products expansions of the Boolean function $F(x,y,z)$ that equals 1 if and only if

**a)** $x = 0$ . . . . . . . . . . $F(x,y,z) = \bar{x}\bar{y}\bar{z} + \bar{x}\bar{y}z + \bar{x}y\bar{z} + \bar{x}yz$
**b)** $xy = 0$ . . . . . . . . . . $F(x,y,z) = \bar{x}\bar{y}\bar{z} + \bar{x}\bar{y}z + \bar{x}y\bar{z} + \bar{x}yz + x\bar{y}\bar{z} + x\bar{y}z$
**c)** $x + y = 0$ . . . . . . . . . . $F(x,y,z) = \bar{x}\bar{y}\bar{z} + \bar{x}\bar{y}z$
**d)** $xyz = 0$ . . . . . . . . . . $F(x,y,z) = \bar{x}\bar{y}\bar{z} + \bar{x}\bar{y}z + \bar{x}y\bar{z} + \bar{x}yz + x\bar{y}\bar{z} + x\bar{y}z + xy\bar{z}$

**Solution:** The expressions above are obtained from the truth tables:

| $x$ | $y$ | $z$ | **a)** |
|---|---|---|---|
| 0 | 0 | 0 | 1 |
| 0 | 0 | 1 | 1 |
| 0 | 1 | 0 | 1 |
| 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | 0 |
| 1 | 1 | 0 | 0 |
| 1 | 1 | 1 | 0 |

| $x$ | $y$ | $z$ | **b)** |
|---|---|---|---|
| 0 | 0 | 0 | 1 |
| 0 | 0 | 1 | 1 |
| 0 | 1 | 0 | 1 |
| 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 1 |
| 1 | 0 | 1 | 1 |
| 1 | 1 | 0 | 0 |
| 1 | 1 | 1 | 0 |

| $x$ | $y$ | $z$ | **c)** |
|---|---|---|---|
| 0 | 0 | 0 | 1 |
| 0 | 0 | 1 | 1 |
| 0 | 1 | 0 | 0 |
| 0 | 1 | 1 | 0 |
| 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | 0 |
| 1 | 1 | 0 | 0 |
| 1 | 1 | 1 | 0 |

| $x$ | $y$ | $z$ | **d)** |
|---|---|---|---|
| 0 | 0 | 0 | 1 |
| 0 | 0 | 1 | 1 |
| 0 | 1 | 0 | 1 |
| 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 1 |
| 1 | 0 | 1 | 1 |
| 1 | 1 | 0 | 1 |
| 1 | 1 | 1 | 0 |

**Exercise 7.** Exercise 6 p. 712 of [1]: Find the sum-of-products expansion that represents a Boolean function $F(x_1, x_2, x_3, x_4, x_5)$ that has the value 1 if an only if three or more of the variables $x_1$, $x_2$, $x_3$, $x_4$ and $x_5$ have the value 1.

**Solution:** The sum-of-product expansion will have 1 term with all 5 terms equal to 1, 5 terms with 4 of the variables equal to 1 and $C(5,10) =$ terms with 3 of the variables equal to 1 :

$$\begin{aligned}
F(x_1,x_2,x_3,x_4,x_5) &= x_1x_2x_3x_4x_5 \\
&+ x_1x_2x_3x_4\bar{x}_5 + x_1x_2x_3\bar{x}_4x_5 + x_1x_2\bar{x}_3x_4x_5 + x_1\bar{x}_2x_3x_4x_5 + \bar{x}_1x_2x_3x_4x_5 \\
&+ x_1x_2x_3\bar{x}_4\bar{x}_5 + x_1x_2\bar{x}_3x_4\bar{x}_5 + x_1\bar{x}_2x_3x_4\bar{x}_5 + \bar{x}_1x_2x_3x_4\bar{x}_5 + x_1x_2\bar{x}_3\bar{x}_4x_5 \\
&+ x_1\bar{x}_2x_3\bar{x}_4x_5 + \bar{x}_1x_2x_3\bar{x}_4x_5 + x_1\bar{x}_2\bar{x}_3x_4x_5 + \bar{x}_1x_2\bar{x}_3x_4x_5 + \bar{x}_1\bar{x}_2x_3x_4x_5
\end{aligned}$$

## References

[1] K. H. Rosen. *Discrete Mathematics and Its Applications*. Mc Graw Hill, 5 edition, 2003.