CS275 GRADED HOMEWORK 7 - PROVISIONAL SOLUTION

GIVE BACK ON TUESDAY NOVEMBER 9TH 2004 AT BEGINNING OF CLASS

For each question, read **each word** with the greatest care and **without hurrying**. If you have doubts about what is asked, **go back** to the wording of the question until the meaning of the question is clear. Then try to find an answer. If you get stuck, don't hesitate to **contact** your T.A. or me.

Please write your section number on your homework as well as a rough estimate of the time you spent solving it.

1. Reminder: Solving recurrence relations

In order to solve the linear homogeneous recurrence relation of degree two

$$x_n = c_1 x_{n-1} + c_2 x_{n-2}, \tag{1.1}$$

you first find the solutions r_1 and r_2 of the characteristic polynomial,

$$r^2 - c_1 r - c_2 = 0.$$

Two cases exist:

a) $r_1 \neq r_2$. Then, all the solutions to Eq. (1.1) take the form

$$x_n = \alpha_1 r_1^n + \alpha_2 r_2^n,$$

where α_1 and α_2 are obtained from the initial conditions, by solving the equations

$$\begin{cases} \alpha_1 + \alpha_2 & = & x_0 \\ r_1\alpha_1 + r_2\alpha_2 & = & x_1. \end{cases}$$

b) $r_1 = r_2$. Then, all the solutions to Eq. (1.1) take the form

$$x_n = (\alpha + \beta n)\, r_1^n,$$

where α and β are obtained from the initial conditions, by solving the equations

$$\begin{cases} \alpha & = & x_0 \\ r_1\alpha + r_2\beta & = & x_1. \end{cases}$$

2. Exercises of Homework 7

Exercise 1. Solve Exercise 8 p. 423 of [1]. A model for the number of lobsters caught per year is based on the assumption that the number of lobsters caught in a year is the average of the number caught in the two previous years.

a) Find a recurrence relation for $\{L_n\}$, where L_n is the number of lobsters caught in year n, under the assumption for this model.
Solution:

$$L_n = \frac{1}{2}(L_{n-1} + L_{n-2}) = \frac{1}{2}L_{n-1} + \frac{1}{2}L_{n-2} \tag{1.1}$$

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b) Find L_n if 100,000 lobsters are caught in year 1 and 300,000 were caught in year 2.
Solution: The characteristic polynomial $r^2 - r/2 - 1/2 = (r-1)(r+1/2)$ has two distinct roots $r_1 = 1$ and $r_2 = -1/2$. Writing $L_n = \alpha_1 r_1^n + \alpha_2 r_2^n$, the initial conditions are

$$\begin{cases} 1e5 = \alpha_1 r_1^1 + \alpha_2 r_2^1 = \alpha_1 - \alpha_2/2 \\ 3e5 = \alpha_1 r_1^2 + \alpha_2 r_2^2 = \alpha_1 + \alpha_2/4, \end{cases}$$

which yield $\alpha_1 = \frac{7}{3}1e5$ and $\alpha_2 = \frac{8}{3}1e5$, so that

$$L_n = 1e5\left(\frac{7}{3} + \frac{8}{3}\left(-\frac{1}{2}\right)^n\right) \simeq 23,333 + 26,667\left(-\frac{1}{2}\right)^n. \tag{1.2}$$

You may also want to check that Eq. (1.1) yields $L_3 = 200,000$, $L_4 = 250,000$, $L_5 = 225,000$ and that Eq. (1.2) yields $L_3 = 1e5\left(\frac{7}{3} + \frac{8}{3}\left(\frac{-1}{8}\right)\right) = 200,000$, $L_4 = 1e5\left(\frac{7}{3} + \frac{8}{3}\left(\frac{1}{16}\right)\right) = 1e5\left(\frac{15}{6}\right) = 250,000$ and $L_4 = 1e5\left(\frac{7}{3} + \frac{8}{3}\left(\frac{-1}{32}\right)\right) = 1e5\left(\frac{27}{12}\right) = 225,000$ too.
Common mistake: not notice that the initial conditions are L_1 and L_2 rather than L_0 and L_1.

Exercise 2. Somewhat like Exercise 4 p. 467 of [1]: Every day at noon, a population census of an aphid[1] colony is done. Every afternoon, each aphid bears two offsprings. Every morning, all aphids older than 24 hours die. On the first day at noon, the colony had just 30 young aphids that had been born that morning.

a) Compute the population of the aphid colony on the 2nd, 3rd, 4th and 5th day at noon.
Solution:
Assuming the initial aphids were born in the morning :
1) 60 aphids are born in the afternoon of day 1, 30 die on the morning of day 2 (since, during the morning, they reach 24 hours of age), so at noon of day 2, there are $x_2 = 60$ aphids.
2) 120 aphids are born in the afternoon of day 2, 60 (those born on day 1) die on the morning of day 3 , so at noon of day 3, there are $x_3 = 120$ aphids.
3) 240 aphids are born in the afternoon of day 3, 120 (those born on day 2) die on the morning of day 4 , so at noon of day 4, there are $x_4 = 240$ aphids.
4) 480 aphids are born in the afternoon of day 4, 240 (those born on day 3) die on the morning of day 5 , so at noon of day 5, there are $x_5 = 480$ aphids.

Assuming the initial aphids were born at noon:
1) 60 aphids are born in the afternoon of day 1, none die on the morning of day 2, so at noon of day 2, there are $x_2 = 90$ aphids.
2) 180 aphids are born in the afternoon of day 2, 90 (those born on day 1) die on the morning of day 3, so at noon of day 3, there are $x_3 = 180$ aphids.
3) 360 aphids are born in the afternoon of day 3, 180 (those born on day 2) die on the morning of day 4, so at noon of day 4, there are $x_4 = 360$ aphids.
4) 720 aphids are born in the afternoon of day 4, 360 (those born on day 3) die on the morning of day 5, so at noon of day 5, there are $x_5 = 720$ aphids.

b) Write a recurrence relation describing the number of aphids in the colony on day n.
Solution: On all days, except the 2nd, the number of births in the afternoon of day n is $2x_n$; the aphids that reach 24 hours and die in the following morning are those that were born on the afternoon of day $n-1$, and they number $2x_{n-1}$. One thus has

$$x_{n+1} = x_n + 2x_n - 2x_{n-1},$$

i.e.

$$x_n = 3x_{n-1} - 2x_{n-2}.$$

Common mistake: write, with no explanation, $x_n = 2x_{n-1}$, which does not correspond to the model.

[1] An aphid is a small insect.

c) Solve this recurrence relation to give a formula for the population on day n.
Solution: The characteristic polynomial is $r^2 - 3r + 2$, which has roots $r_1 = 2$ and $r_2 = 1$.

Assuming the initial aphids were born in the morning : The recurrence relation holds from the start, and one can use the initial conditions $x_1 = 30$, $x_2 = 60$ to obtain

$$x_n = 30 \cdot 2^{n-1} \text{ for } n \geq 1.$$

Assuming the initial aphids were born at noon : The recurrence relation holds from day 3, and one can use the initial conditions $x_2 = 90$, $x_3 = 90$ to obtain

$$x_n = 45 \cdot 2^{n-1} \text{ for } n \geq 2.$$

Note: answers that are wrong, but coherent with the answer given in question **b)** were counted as correct.

d) On what day will the population reach 100,000?
Solution: In both cases, in the afternoon of the 12th day, the population will reach $3 \cdot x_{12}$, which is greater than 100,000. The first day n for which the $x_n \geq 100,000$ is $n = 13$.
Note: answers $n = 12$ and $n = 13$ were counted as correct as long as they were coherent with the answer given in questions **b)** and **c)** were counted as correct.

Exercise 3. Solve the recurrence relation $x_n = 2x_{n-2} - x_{n-4}$, when $x_0 = 2$, $x_1 = 1$, $x_2 = 2$, $x_3 = \frac{3}{2}$.

Hint: Solve the recurrence relation separately for n odd and for n even. As usual, compute the first few terms of the sequence, to see what it looks like.

Solution: Define $y_n = x_{2n}$ and $z_n = x_{2n+1}$. One has $y_n = 2y_{n-1} - y_{n-2}$, $y_0 = y_1 = 2$ and $z_n = 2z_{n-1} - z_{n-2}$, $z_0 = 1$, $z_1 = \frac{3}{2}$. The characteristic polynomial is $r^2 - 2r + 1 = (r-1)^2$, which has a single (double) root $r_1 = 1$. From the initial conditions, one has

$$y_n = 2 + 0 \cdot n = x_{2n}$$

and

$$z_n = 1 + \frac{1}{2}n = x_{2n+1}.$$

Putting things back together, one gets

$$x_n = \begin{cases} 2 & \text{if } n \text{ is even,} \\ \frac{3+n}{4} & \text{if } n \text{ is odd.} \end{cases}$$

To obtain $\frac{3+n}{4}$, note that, if n is odd, then $n = 2m + 1$ for some $m = \frac{n-1}{2}$ and thus $x_n = z_m = 1 + \frac{m}{2} = 1 + \frac{n-1}{4} = \frac{3+n}{4}$.

References

[1] K. H. Rosen. *Discrete Mathematics and Its Applications.* Mc Graw Hill, 5 edition, 2003.