# CS275 GRADED HOMEWORK 8 - SOLUTION

GIVE BACK ON TUESDAY NOVEMBER 16TH 2004 AT BEGINNING OF EXAM

For each question, read **each word** with the greatest care and **without hurrying**. If you have doubts about what is asked, **go back** to the wording of the question until the meaning of the question is clear. Then try to find an answer. If you get stuck, **contact** your T.A. or me.

Advertisement: Could you benefit from mentoring? See `www.mentornet.net` if you feel a little extra guidance could help in your studies.

Please write your section number and an estimate of the time you spent on this HW.

## 1. Reminders

1.1. **Graph definitions.** Recall that a graph is an object $(V, E)$, where $V$ is a set of vertices and $E$ is a set of edges. A weighed graph is an object $(V, E, w)$, where $w$ is a weight function defined on pairs of vertices. You should know each of the definitions below and be able to find examples of such objects.

- Simple, directed, weighed, labeled, multi- and pseudo- graphs.
- Nouns: Edge, vertex, endpoint; degree, in- and out- degree of an edge.
- Adjectives: adjacent, incident, initial, final (vertex), incident to/from, isolated, pendant.
- More nouns: path, circuit. Subgraph, union of graphs. Connected component, cut edges/vertices.
- More adjectives: connected, strongly/weakly connected graph, simple path/circuit.
- Properties: Handshaking theorem, theorems 2 and 3 of [2].

1.1.1. *Example.* In a software project, the dependencies between source files are:

- File `main.c` has dependencies on `stdio.h`, `mylib.h` and `mylib.c`.
- File `mylib.c` has dependencies on `stdio.h` and `mylib.h`.
- File `mylib.h` has dependencies on `stdio.h`.

To represent the dependency relation between files, using mathematical notation, one could define the a directed graph with vertices $V = \{\texttt{main.c}, \texttt{mylib.c}, \texttt{mylib.h}, \texttt{stdio.h}\}$, and edges

$$\begin{aligned} E \;=\; & \{(\texttt{main.c}, \texttt{mylib.c}), (\texttt{main.c}, \texttt{mylib.h}), (\texttt{main.c}, \texttt{stdio.h}), \\ & (\texttt{mylib.c}, \texttt{mylib.h}), (\texttt{mylib.c}, \texttt{stdio.h}), \\ & (\texttt{mylib.h}, \texttt{stdio.h})\}. \end{aligned}$$

Note that, since $E$ is a directed graph rather than a simple graph, each element of $E$ is an ordered pair $(f, g)$, rather than a 2-element set $\{f, g\}$.

## 2. Exercises

**Exercise 1.** Extended version of Exercise 24 p. 555 of [2]. For which values of $n$ are these graphs bipartite?

**(a)** $K_n$ **(b)** $C_n$ **(c)** $W_n$ **(d)** $Q_n$ **(e)** $S_n$

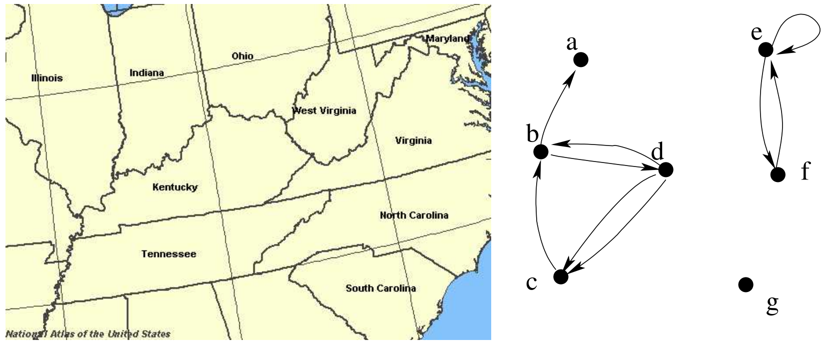
where $S_n$ is the "star" simple graph $(\{0, \ldots, n\}, \{\{0,1\}, \{0,2\}, \{0,3\}, \ldots, \{0,n\}\})$.

**Solution:**

**a)** $K_n$: $n \in \{1, 2\}$ (ok if only 2 is listed).

**b)** $C_n$: even $n$. "Parties" are $\{1, 3, \ldots, n-1\}$, and $\{2, 4, \ldots, n\}$.

**c)** $W_n$ is never bipartite.




**(a)** **(b)**

FIGURE 2.1. **(a)** Map for Ex. 3. Source: Natl. Atlas of the U.S.A. [1]. **(b)** Graph for Ex. 4

**d)** $Q_n$ is bipartite for all $n$. Let $V = \{X = (x_1, \ldots, x_n) \mid \forall i \in \{0, \ldots, n\}, x_i \in \{0,1\}\} = \{0,1\}^n$ be the set of vertices of $Q_n$. The "parties" are then $\{X \in V \mid \sum x_i \text{ is even}\}$ and $\{X \in V \mid \sum x_i \text{ is odd}\}$.

**e)** $S_n$ is bipartite for all $n$. The "parties" are the "center" vertex $\{0\}$ and all the other vertices $\{1, \ldots n\}$.

**Exercise 2.** Solve Exercise 34 p. 556 of [2]. Let $G$ be a graph with $v$ vertices and $e$ edges. Let $M$ be the maximum degree of the vertices of $G$, and let $m$ be the minimum degree of the vertices of $G$. Show that

**a)** $2e/v \geq m$.
**Solution:** Let $\{X_1, \ldots, X_v\}$ be the set of vertices. The handshaking theorem says that $2e = \sum_{i=1}^{v} \deg(X_i)$. By definition of $m$, one has $\forall i \in \{1, \ldots, v\}, \deg(X_i) \geq m$, so that $\sum_{i=1}^{v} \deg(X_i) \geq \sum_{i=1}^{v} m = mv$. Joining the two together, one gets $2e \geq mv$, so that $\frac{2e}{v} \geq m$.

**b)** $2e/v \leq M$.
**Solution:** Let $\{X_1, \ldots, X_v\}$ be the set of vertices. By definition of $M$, one has $\forall i \in \{1, \ldots, v\}, \deg(X_i) \leq M$, so that $\sum_{i=1}^{v} \deg(X_i) \leq \sum_{i=1}^{v} M = Mv$. By using the handshaking theorem as above, one gets $2e \leq Mv$ and thus $M \geq \frac{2e}{v}$.

**Exercise 3.** Consider the states of Kentucky (KY), Tennessee (TN), North Carolina, Virginia (VA), West Virginia (WV), North (NC) and South Carolina (SC), Ohio (OH), Indiana (IN) and Illinois (IL), represented in the map in Figure 2.1. We want to build a graph whose vertices are state names and where an edge connects two states if and only if the states share a common border.

**a)** Which type of graph (simple, directed, pseudo-) is most appropriate to represent this data?
**Solution:** A simple graph will do.

**b)** Write the graph in mathematical notation.
**Solution:** {{IL,IN}, {IL,KY}, {IN,KY}, {IN,OH}, {OH,KY}, {OH,WV}, {KY,WV}, {KY,VA}, {KY,TN}, {WV,VA}, {VA,NC}, {VA,TN}, {NC,TN}, {NC,SC}}

**Exercise 4.** In the directed multi-graph of Figure 2.1, right. Write down the following elements:

**a)** Vertices, edges.
**Solution:** $V = \{a, b, c, d, e, f, g\}$, writing e.g. $ba$ for $(b, a)$, one has $E = \{ba, bd, cb, db, dc, ee, ef, fe\}$.

**b)** Multiplicity of the edges.
**Solution:** The multiplicities are: $w(dc) = 2$ and $w(e) = 1$ for all other edges in $e \in E$.

**c)** Isolated and pendant vertices.
**Solution:** Isolated: $g$. Pendant $a$.

**d)** In-degree, out-degree and degree of each vertex.

**Solution:**

| | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | $g$ |
|---|---|---|---|---|---|---|---|
| $\deg^-$ | 1 | 2 | 2 | 1 | 2 | 1 | 0 |
| $\deg^+$ | 0 | 2 | 1 | 3 | 2 | 1 | 0 |
| $\deg$ | 1 | 4 | 3 | 4 | 4 | 2 | 0 |

**e)** Strongly and weakly connected components of the graph.
**Solution:** Strongly connected components: $\{b, c, d\}$ and $\{e, f\}$. Weakly connected components: $\{a, b, c, d\}$ and $\{e, f\}$.

**Exercise 5.** In Fig. 2.1 **(b)**, say whether the following objects can be found and, if yes, describe one.

**a)** A simple path of length 3 starting in $d$ and ending in $a$.
**Solution:** Yes: $(d, c, b, a)$.

**b)** A path passing through $a$, $b$, $c$ and $d$ in any order.
**Solution:** No: Such a path would have to end in $a$ and thus not go "through" $a$ (using the definition of the book). Saying yes: $(d, c, b, a)$ is also ok, though.

**c)** A path of length 4 starting in $d$ and ending in $a$.
**Solution:** $(d, b, d, b, a)$.

**d)** A circuit starting at $d$, and passing through $a$.
**Solution:** No. A path passing through $a$ necessarily terminates there.

**e)** A circuit of length three starting in $e$.
**Solution:** Yes: e.g. $(e, e, e, e)$,$(e, f, e, e)$..

## References

[1] National atlas of the U.S.A. http://nationalatlas.gov.
[2] K. H. Rosen. *Discrete Mathematics and Its Applications.* Mc Graw Hill, 5 edition, 2003.