CS275 GRADED HOMEWORK 9 - SOLUTION

GIVE BACK ON TUESDAY NOVEMBER 30TH 2004 AT BEGINNING OF CLASS

For each question, read **each word** with the greatest care and **without hurrying**. If you have doubts about what is asked, **go back** to the wording of the question until the meaning of the question is clear. Then try to find an answer. If you get stuck, **contact** your T.A. or me.

Please write your section number and an estimate of the time you spent on this HW.

1. Reminders

1.1. Graph isomorphisms.

Definition. Isomorphic graphs. Let $G = (V, E)$ and $G' = (V', E')$ be simple graphs. G and G' are said to be isomorphic if and only if and only if there exists a bijection $f : V \longrightarrow V'$ such that

$$\forall u, v \in V, \quad \{u, v\} \in E \iff \{f(u), f(v)\} \in E'.$$

This definition is valid for graphs with loops too. In the case of multi-graphs (as defined in the helper page `http://www.vis.uky.edu/~etienne/cs275/helpers.html#graphs`), we will use the definition:

Definition. Isomorphic weighed/multi- graphs. Let $G = (V, E, w)$ and $G' = (V', E', w)$ be multi-graphs. G and G' are said to be isomorphic if and only if and only if there exists a bijection $f : V \longrightarrow V'$ such that

$$\forall u, v \in V, \ w(\{u, v\}) = w(\{f(u), f(v)\}).$$

Isomorphic directed graphs are likewise defined by replacing unordered pairs by ordered pairs, i.e. by replacing the curly brackets $\{\}$ with parentheses ().

Definition. Graph Isomorphism. The function f in the definitions above is called a graph isomorphism from G to G' or between G and G'.

Proposition. *If f is a graph isomorphism between two graphs $G = (V, E)$ and $G' = (V', E')$, then*

- **a)** The number of vertices is preserved: $|V| = |V'|$.
- **b)** The number of edges is preserved: $|E| = |E'|$.
- **c)** The degree of vertices is preserved. For all $u \in V$, one has $\deg(u) = \deg(f(u))$. In the case of directed graphs, one also has $\deg^+(u) = \deg^+(f(u))$ and $\deg^-(u) = \deg^-(f(u))$.
- **d)** In weighed graphs (multi-graphs), the multiplicity of vertices is preserved : for all $u, v \in V$, one has $w(\{u, v\}) = w(\{f(u), f(v)\})$.
- **e)** Paths and circuits are preserved: the image of a path (resp. circuit) in G is a path (resp. circuit) in G' with same length.
- **f)** For any number $d \in \mathbb{N}$, the number of vertices of G with degree d is the same as the number of vertices of G' with degree d. That is, in mathematical notation:

$$\forall d \in \mathbb{N}, |\{v \in V \mid \deg(v) = d\}| = |\{v' \in V' \mid \deg(v') = d\}|.$$

The same holds for the in- and out- degrees of a directed graph.

1.2. Trees.

Definition. Full m-ary tree. An m-ary tree T if full if and only if each vertex has 0 or m children.

Definition. Balanced rooted tree. A tree T with height (depth) h is balanced if and only if each leaf is at level h or $h - 1$.

Definition. Complete rooted m-ary tree. A rooted m-ary tree T is complete if and only if it is full and all leaves are at the same level.

2. Exercises

Exercise 1. Let G, G' and G'' be graphs such that G is isomorphic with G' and G' is isomorphic with G''. Let f_1 (resp. f_2) be a graph isomorphism between G and G' (resp. G' and G'').

a) Prove that G is isomorphic with itself.
Solution: Let $G = (V, E)$. It is easy to show that the identity function $I : v \in V \longrightarrow v \in V$ is a graph isomorphism.

b) Prove that G' is isomorphic with G, e.g. by showing that f_1^{-1} is a graph isomorphism.
Solution: Let $G' = (V', E')$, $\{u', v'\} \in E'$ be an edge of G', $u = f_1^{-1}(u')$ and $v = f_1^{-1}(v')$. By definition of f_1^{-1}, one has $u' = f_1(u)$ and $v' = f_1(v)$. Since f_1 is a graph isomorphism, one has $\{u, v\} \in E \iff \{f_1(u), f_1(v)\} \in E'$ and thus $\{f_1^{-1}(u'), f_1^{-1}(v')\} \in E \iff \{u', v'\} \in E'$ and thus f_1^{-1} is also a graph isomorphism.

c) Prove that G is isomorphic with G'', e.g. by showing that $f_2 \circ f_1$ is a graph isomorphism.
Solution: Let $G'' = (V'', E'')$. Since f_1 is an isomorphism, $\{u, v\} \in E \iff \{f_1(u), f_1(v)\} \in E'$; similarly, $\{f_1(u), f_1(v)\} \in E' \iff \{f_2(f_1(u)), f_2(f_1(v))\} \in E''$, so that indeed, $\{u, v\} \in E \iff \{f_2(f_1(u)), f_2(f_1(v))\} \in E''$ and $f_2 \circ f_1$ is an isomorphism.

Exercise 2. Let R be the relation defined on pairs of graphs.

$$R(G, G') \equiv G \,\text{and}\, G' \,\text{are isomorphic.}$$

a) Show that R is an equivalence relation (you may want to use results from Ex. 1).
Solution: Exercise 1 a-c prove, respectively that R is reflexive, symmetric ant transitive.

b) Consider the 25 graphs A, B, etc in Figure 2.1. Using the properties listed in the "Reminder" section above, find a sufficient reason for which
1) A is not isomorphic to B. . . . . A has 5 vertices, while B has 3.
2) A is not isomorphic to F. . . . . A has two vertices of degree 1 while F has three.
3) H is not isomorphic to E. . . . . . E has three vertices of degree one, while H has four.

c) Let $\mathcal{L}$ be the set of the 25 graphs A, B, etc in Figure 2.1. Write the equivalence classes in $\mathcal{L}$ for the relation R. I.e. find the subsets of $\mathcal{L}$ that contain isomorphic graphs.
Solution: $\{A, R\}$, $\{B\}$,$\{C, S, U\}$,$\{D\}$,$\{E\}$,$\{F\}$, $\{G, T, Y\}$, $\{H, K\}$,$\{I, J\}$, $\{L, V\}$, $\{M, W\}$, $\{N, Z\}$, $\{P\}$, $\{Q\}$, $\{X\}$.
Common mistake: forget the classes that only have one element.

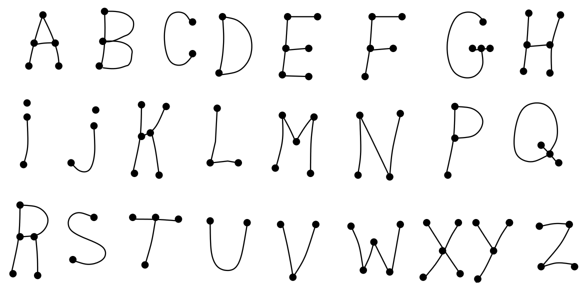
Exercise 3. Recall the definitions of Hamilton and Euler circuits, of the complete graph K_n and of the complete bipartite graph $K_{m,n}$.

a) Write the number of Hamilton circuits in K_n as a function of n.
Solution: A Hamilton circuit in K_n may traverse each of the vertices in any order. Since there are $n!$ permutations of $\{1, \ldots, n\}$, there are $n!$ Hamilton circuits in K_n.
Another valid answer is $(n-1)!$, which is the number of distinct paths, if one considers that two paths are "equal"[1] if they only differ by their starting points.

b) For what values of m, n does $K_{m,n}$ have an Euler circuit?
Solution: $K_{m,n}$ has an Euler circuit iff each vertex has even degree, i.e. iff m and n are both even.

c) For what values of m, n does $K_{m,n}$ have a Hamilton circuit?
Solution: For $m = n$. Let $V_1 = \{u_1, \ldots, u_m\}$ and $V_2 = \{v_1, \ldots, v_n\}$ be the two "parties" of vertices of $K_{m,n}$. A circuit in $K_{m,n}$ is of the form $(u_{i_1}, v_{j_1}, u_{i_2}, \ldots, u_{i_L}, v_{j_L}, u_{i_1})$ and thus passes through as many u_is as v_is. Since, in a Hamilton circuit, all the traversed u_i and v_i must be distinct and must cover V_1 and V_2, one has $L = m = n$ and thus $m = n$.

[1] Equivalent, for some equivalence relation that could have been specified.



These drawings of graphs could be represented by the weighed pseudo-graphs:

- A: Vertices: $V = \{1, \ldots, 5\}$, e.g. numbered top to bottom and left to right. Edges: $E = \{\{1,2\}, \{1,3\}, \{2,3\}, \{2,4\}, \{3,5\}\}$. Multiplicity: $w(e) = 1$ for all edge e.
- B : $V = \{1,2,3\}$. $E = \{\{1,2\}, \{2,3\}\}$. $w(e) = 2$ for all edge e.
- C : $V = \{1,2\}$. $E = \{\{1,2\}\}$. $w(e) = 1$ for all edge e.
- D : $V = \{1,2\}$. $E = \{\{1,2\}\}$. $w(e) = 2$ for all edge e.
- etc...

Notice e.g. that C and D differ only by the multiplicity of their edge.

FIGURE 2.1. Graphs for Exercise 2.

Exercise 4. Solve Exercises 28 and 30 of p. 643 [1].

a) How many vertices and how many leaves does a complete m-ary tree of height h have?
Solution: m^h. Proof by induction. Let $P(h)$ be the proposition "all complete m-ary trees of depth h have m^h leaves."
Basis step: The only (up to isomorphism) complete m-ary tree of height 0 is $T = (\{0\}, \emptyset)$, which has $1 = m^0$ leaves.
Induction step: Assume $P(D)$ holds. Let T be a complete m-ary tree of height $D+1$. Let $u_1, \ldots, u_N$ be its leaves. Let $v_1, \ldots, v_Q$ be the parents of the leaves. Since each of the v_i has m children, one has $N = mQ$. Let T' be T without its leaves and the corresponding edges. It is clear that T' is complete and has depth D. Since $P(D)$ is true, T' has $Q = m^D$ leaves and thus $N = mQ = m \cdot m^D = m^{D+1}$.

b) Show that a full m-ary balanced tree of height h has more than m^{h-1} leaves.
Solution: Assume $m > 1$ - otherwise, full balanced m-ary trees of depth h have just $1 \not> 1^{h-1}$ leaf and the proposition is false.
Let T' be T without its leaves at depth D and the corresponding edges. T' is complete and has depth D, so that, by **a)**, it has m^D leaves. If $p \geq 1$ is the number of leaves of T' that are internal nodes of T, these nodes are parents of mp leaves of T, which thus has $m^D + mp - p > m^D$ leaves.

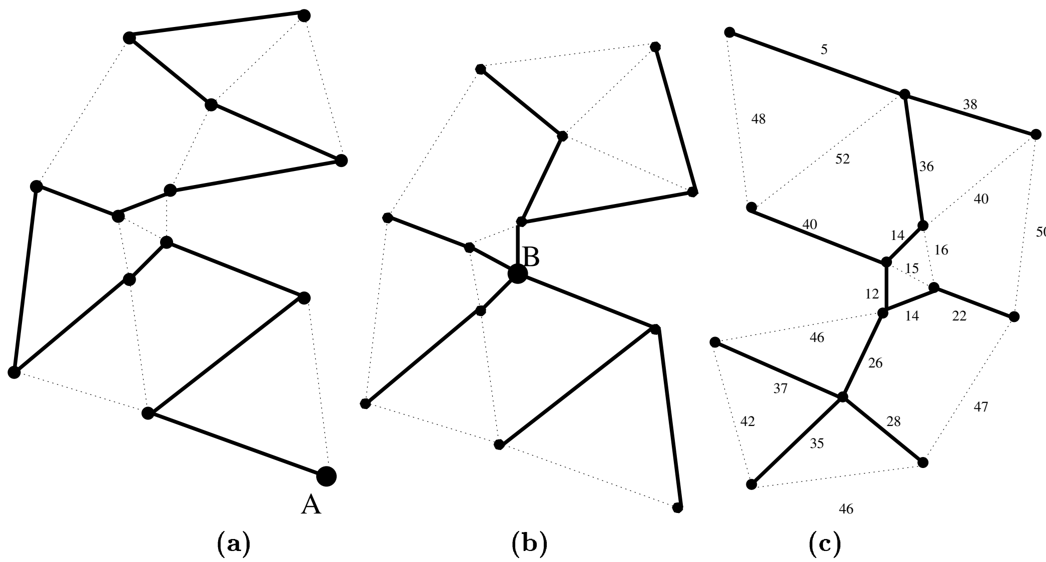


FIGURE 4.1. Graphs for Exercise 5.

Exercise 5. Consider the graphs drawn in Figure 4.1.

a) Draw a spanning tree of the graph in Figure 4.1 (a), rooted in A.
b) Draw a spanning tree of the graph in Figure 4.1 (b), rooted in B.
c) Draw a minimal spanning tree of the graph in Figure 4.1 (c).
d) What is the total length of the edges in this minimal spanning tree? 307

Solution: See Figure 4.1.

REFERENCES

[1] K. H. Rosen. *Discrete Mathematics and Its Applications.* Mc Graw Hill, 5 edition, 2003.