# DISCRETE MATH CS275 MIDTERM EXAM 1

TUESDAY OCT 5TH 2004, 12:30 - 1:45PM.

Check that you have 4 exercises on 2 pages.
No calculator is needed or allowed in this exam.
When asked a cardinal, it is enough to give a formula.

Advice:

- Read all the exercises before answering any.
- Spend 10 minutes on each exercise, then return to questions you missed.
- For each exercise, read **each word** with the greatest care and **without hurrying**. Read it **many times** if needed, until the meaning of the questions is clear.

**Exercise 1.** Let $A$ and $B$ be finite sets, with cardinals $m$ and $n$, respectively, and let $p$, $q$ be natural numbers. Write the cardinals of the following sets[1]

**a)** $A \cup B$

**b)** $A \times B$

**c)** $\overbrace{A \times \ldots \times A}^{p}$

**d)** $\left\{C \,\middle|\, C \subseteq A,\ |C| = p\right\}$

**e)** $\left\{(x_1, \ldots, x_p) \,\middle|\, \forall i,\ x_i \in A\right\}$

**f)** $\left\{(x_1, \ldots, x_p) \,\middle|\, \forall i,\ x_i \in \mathbb{N},\ \sum_{i=0}^{p} x_i = q\right\}$

**g)** $\left\{(x_1, \ldots, x_p) \,\middle|\, \forall i,\ x_i \in A,\ \forall j \neq i,\ x_i \neq x_j\right\}$

**h)** The set of combinations (i.e. subsets) of $p$ elements of $A$.

**i)** The set of permutations (without repetition) of $p$ elements of $A$.

**Exercise 2.** One day, a programmer is given a program to debug and improve. Let $x_n$ be the number of bugs in the program at the beginning of day $n$. The program originally has $x_0 = 15$ bugs.

- Each morning, the programmer does debugging and removes $\left\lceil \frac{1}{2} x_n \right\rceil$ bugs from the program.
- In the afternoon, the programmer adds features to the program, but, in the process, adds 4 new bugs.

Questions:

**a)** Write $x_{n+1}$ as a function of $x_n$.
**b)** Compute $x_1$, $x_2$, $x_3$, $x_4$, $x_{365}$ and $x_{365{,}000}$.
**c)** Justify your answer to the last question with a proof by induction.

[1] Not by just writing $|.|$ around each set.

**Exercise 3.** Write in English, and then either prove or disprove, each of the following statements. You may keep arithmetic expressions such as $\frac{1}{10}n^2$ and $(n-2)^2$ in your English sentences.

**a)** $\exists n \in \mathbb{N},\ \frac{1}{10}n^2 \in \mathbb{N}$.

**b)** $\exists! n \in \mathbb{Z},\ (n-2)^2 = 0$.

**c)** $\exists A\, \forall n\, (n \in \mathbb{N} \Longrightarrow n \in A)$.

**d)** $\forall A\, \forall B,\ A \subseteq B \implies |A| < |B|$.

**e)** $\forall A\, \forall B,\ |A| + |B| = |A \cup B| \Longrightarrow A \cap B = \emptyset$.

*Reminders:*

- To prove a "there exists an $X$ ..." statement, it is sufficient to find an object $X$ that verifies the property "...".
- To prove a "for all $X$, $Y$ ..." statement, you usually start by "let $X$, $Y$ be ..." and then showing that $X$, $Y$ verify the desired property.
- To disprove a statement, you may prove its negation. The negation of "for all" is "there exists." The negation of "there exists" is "for all."

**Exercise 4.** A man has the following clothing:

- 12 different T-shirts.
- 3 different pairs of pants.
- 2 different pairs of shoes.
- 10 different pairs of socks.
- 10 different pairs of underpants.

**a)** When getting dressed, he chooses one T-shirt, one pair of shoes, of pants, of socks and of underpants. In how many different ways can he dress? Justify your answer.

**b)** One day, he goes on a short trip. He picks

1) two T-shirts,
2) two pairs of socks and
3) two pairs of underpants,

and puts them in his bag (order does not matter). In how many different ways can he make his choice? Justify your answer.

**c)** On his way, he gets two different books from a bookstore, which has 1000 different books for sale, and puts them in his bag (order does not matter). In how many different ways can he choose two books out of 1000? Justify your answer.