DISCRETE MATH CS275 MIDTERM EXAM 1 : SOLUTIONS

TUESDAY OCT 5TH 2004, 12:30 - 1:45PM.

Advice:

- Read all the exercises before answering any.
- Spend 10 minutes on each exercise, then return to questions you missed.
- For each exercise, read **each word** with the greatest care and **without hurrying**. Read it **many times** if needed, until the meaning of the questions is clear.

Exercise 1. Let A and B be finite sets, with cardinals m and n, respectively, and let p, q be natural numbers. Write the cardinals of the following sets[1]

a) $A \cup B$ .... $m + n - |A \cap B|$

b) $A \times B$ .... mn

c) $\overbrace{A \times ... \times A}^{p}$ .... m^p

d) $\left\{C \,\middle|\, C \subseteq A,\ |C| = p\right\}$ .... $C(m,p)$ because $\left\{C \,\middle|\, C \subseteq A,\ |C| = p\right\}$ is
the set of subsets of A that have cardinal p, i.e.
the set of subsets of p elements out of the m elements of A i.e.
the set of combinations of p elements out of the m elements of A.
Common mistake: Not recognize that C -a subset- is a combination.

e) $\left\{(x_1, ..., x_p) \,\middle|\, \forall i,\ x_i \in A\right\}$ .... m^p because $\left\{(x_1, ..., x_p) \,\middle|\, \forall i,\ x_i \in A\right\} = \underbrace{A \times \ldots \times A}_{p}$ is the set of sequences of p elements of A. For example, with $m = 4$, $A = \{1..4\}$ and $p = 3$, elements of $A \times A \times A$ would be $(1,1,1)$, ..., $(1,1,4)$, $(1,2,1)$, ... $(4,4,3)$, $(4,4,4)$.
Common mistake: Not recognize that $(x_1, ..., x_p)$ is an element of A^p.

f) $\left\{(x_1, ..., x_p) \,\middle|\, \forall i,\ x_i \in \mathbb{N},\ \sum_{i=0}^{p} x_i = q\right\}$ .... $C(q+p-1, p-1) = C(p+q-1, q)$.
This is the number of ways of catching a total of q fishes in a pond where p different types of fishes live (x_i is the number of caught fishes of type i).

g) $\left\{(x_1, ..., x_p) \,\middle|\, \forall i,\ x_i \in A,\ \forall j \neq i,\ x_i \neq x_j\right\}$ .... $P(m,p)$ because $\left\{(x_1, ..., x_p) \,\middle|\, \forall i,\ x_i \in A,\ \forall j \neq i,\ x_i \neq x_j\right\}$ is
the set of sequences $(x_1, ..., x_p)$ of p distinct elements of A, i.e.,
the set of permutations (without repetition) of p distinct elements of A.
Common mistake: Not recognize that a sequence of distinct elements is a permutation.

h) The set of combinations (i.e. subsets) of p elements of A. .... $C(m,p)$

i) The set of permutations (without repetition) of p elements of A. . $P(m,p)$

[1] Not by just writing $|.|$ around each set.

Exercise 2. One day, a programmer is given a program to debug and improve. Let x_n be the number of bugs in the program at the beginning of day n. The program originally has $x_0 = 15$ bugs.

- Each morning, the programmer does debugging and removes $\left\lceil \frac{1}{2} x_n \right\rceil$ bugs from the program.
- In the afternoon, the programmer adds features to the program, but, in the process, adds 4 new bugs.

Questions:

a) Write x_{n+1} as a function of x_n.
Solution:

$$x_{n+1} = x_n - \left\lceil \frac{1}{2} x_n \right\rceil + 4 = \left\lfloor \frac{1}{2} x_n \right\rfloor + 4.$$

Common mistake: write too quickly $x_{n+1} = \left\lceil \frac{1}{2} x_n \right\rceil + 4$ which is wrong.

b) Compute x_1, x_2, x_3, x_4, x_{365} and $x_{365,000}$.
Solution:

$$\begin{aligned}
x_1 &= \left\lfloor \frac{1}{2} 15 \right\rfloor + 4 = \lfloor 7.5 \rfloor + 4 = 7 + 4 = 11, \\
x_2 &= \left\lfloor \frac{1}{2} 11 \right\rfloor + 4 = \lfloor 5.5 \rfloor + 4 = 9, \\
x_3 &= \left\lfloor \frac{1}{2} 9 \right\rfloor + 4 = 8, \\
x_4 &= \left\lfloor \frac{1}{2} 8 \right\rfloor + 4 = 8, \\
&\vdots \\
x_{365} &= \left\lfloor \frac{1}{2} 8 \right\rfloor + 4 = 8 = x_{365,000}.
\end{aligned}$$

c) Justify your answer to the last question with a proof by induction.
Solution: Let $P(n)$ be the proposition $x_n = 8$.
Basis step: $x_3 = 8$.
Induction step: Suppose that $P(n)$ is true, i.e. that $x_n = 8$. Then, $\left\lfloor \frac{1}{2} x_n \right\rfloor = 4$ and $\left\lfloor \frac{1}{2} x_n \right\rfloor + 4 = 8$, so that $x_{n+1} = 8$ and $P(n+1)$ is thus true.
Common mistake: Take $P(n) \equiv x_{n+1} = \left\lfloor \frac{1}{2} x_n \right\rfloor + 4$. This equation is not the result of the last question, but of the question-before-last. The equation $x_{n+1} = \left\lfloor \frac{1}{2} x_n \right\rfloor + 4$ is the way you model the evolution of the number of bugs. You obtained this expression by translating the problem from English into a mathematical expression. If your translation is correct, your model should be correct and you can use the *assumption* $x_{n+1} = \left\lfloor \frac{1}{2} x_n \right\rfloor + 4$ to prove that $x_n = 8$ for $n \geq 3$.

Exercise 3. Write in English, and then either prove or disprove, each of the following statements. You may keep arithmetic expressions such as $\frac{1}{10}n^2$ and $(n-2)^2$ in your English sentences.

a) $\exists n \in \mathbb{N},\ \frac{1}{10}n^2 \in \mathbb{N}$.

Solution:
There exists a natural number n s.t. $n^2/10$ is a natural number.
This statement is true. Let $n = 10$. Then, $n^2/10 = 100/10 = 10 \in \mathbb{N}$.

b) $\exists! n \in \mathbb{Z},\ (n-2)^2 = 0$.

Solution: There is a unique integer such that $(n-2)^2 = 0$.
This statement is true. First, $n = 2$ is an integer that solves the desired equation. Then, for any $n \in \mathbb{Z}$, $(n-2)^2 = 0$ implies that $n - 2 = 0$, i.e. that $n = 2$, so that $n = 2$ is the only solution.

c) $\exists A\, \forall n\, (n \in \mathbb{N} \Longrightarrow n \in A)$.

Solution: There is a set A s.t. for all n, if n belongs to $\mathbb{N}$, then n belongs to A. This is the same as
There is a set A s.t. all n belonging to $\mathbb{N}$ belongs to A. i.e.
There is a set A s.t. all natural numbers belong to A.
This statement is true. Take $A = \mathbb{N}$. Then indeed, for any n, one has $n \in \mathbb{N} \Longrightarrow n \in A$.

d) $\forall A\, \forall B,\ A \subseteq B \implies |A| < |B|$.

Solution: For all sets A and B, if A is included in B, then the cardinal of A is strictly smaller than that of B.
This statement is false. Let $A = B$. Then, $A \subseteq B$ and $|A| = |B|$ and thus $\neg\,(|A| < |B|)$.
Common mistakes: Do as if $\subset$ or $\leq$ was written in place of $\subseteq$ and $<$.

e) $\forall A\, \forall B,\ |A| + |B| = |A \cup B| \Longrightarrow A \cap B = \emptyset$.
Solution: For all A and B, if the sum of the cardinals of $A$and B is the cardinal of the union of A and B, then the intersection of $A \cap B$ is empty.
This statement is true. Let A and B be two sets s.t. $|A|+|B| = |A \cup B|$. We will show that this implies that $A \cap B = \emptyset$. Since, by the inclusion/exclusion principle, $|A \cup B| = |A| + |B| - |A \cap B|$, one has $|A \cap B| = |A| + |B| - |A \cup B| = 0$, so that $A \cap B = \emptyset$.
Common mistake: Say or show that $A \cap B = \emptyset$ implies $|A|+|B| = |A \cup B|$, while it is the converse, $|A| + |B| = |A \cup B| \Longrightarrow A \cap B = \emptyset$, which should be proved.
Common mistake: Take two set A and B, e,g, "$A = \{1, 2, 3\}$, $B = \{...\}$" and show that the statement is true.
Note: This is only true for finite sets.

Reminders:

- To prove a "there exists an X ..." statement, it is sufficient to find an object X that verifies the property "...".
- To prove a "for all X, Y ..." statement, you usually start by "let X, Y be ..." and then showing that X, Y verify the desired property.

- To disprove a statement, you may prove its negation. The negation of "for all" is "there exists." The negation of "there exists" is "for all."

Exercise 4. A man has the following clothing:

- 12 different T-shirts.
- 3 different pairs of pants.
- 2 different pairs of shoes.
- 10 different pairs of socks.
- 10 different pairs of underpants.

a) When getting dressed, he chooses one T-shirt, one pair of shoes, of pants, of socks and of underpants. In how many different ways can he dress? Justify your answer.
Solution: $12 \cdot 3 \cdot 2 \cdot 10 \cdot 10 = C(12,1) \cdot C(3,1) \cdot C(2,1) \cdot C(10,1) \cdot C(10,1)$.
A way of dressing can be defined by a sequence $(t, p, s, x, u) \in \{1..12\} \times \{1..3\} \times \{1..2\} \times \{1..10\} \times \{1..10\}$, where t is the number of the worn T-shirt, p is the number of the worn pair of pants etc.

b) One day, he goes on a short trip. He picks

1) two T-shirts,
2) two pairs of socks and
3) two pairs of underpants,

and puts them in his bag (order does not matter). In how many different ways can he make his choice? Justify your answer.

Solution: $C(12,2) \cdot C(10,2) \cdot C(10,2)$.
There $C(12,2)$ ways of choosing two T-shirts out of 12, $C(10,2)$ ways of choosing two pairs of socks out of 10 and $C(10,2)$ ways of choosing 2 out of 10 pairs of underpants.

c) On his way, he gets two different books from a bookstore, which has 1000 different books for sale, and puts them in his bag (order does not matter). In how many different ways can he choose two books out of 1000? Justify your answer.
Solution: $C(1000,2)$.