DISCRETE MATH - CS275 MIDTERM 2

NOVEMBER 16TH 2004, 12:30-13:45

Each of the 4 exercises is worth 5 points.
No calculator is needed or allowed in this exam.
All answers must come with an explanation.

Advice:

- Read all the exercises before answering to any.
- Spend 10 to 15 minutes on each exercise, then return to questions you missed.
- For each exercise, read **each word** with the greatest care and **without hurrying**. If needed, read it **many times**, until the meaning of the questions is clear.
- Check for extra information on the blackboard.

Exercise 1. Let F, G and H be Boolean functions of degree four, defined by

$$\begin{aligned} F(w,x,y,z) &= w+x+y+z \\ G(w,x,y,z) &= \bar{w}xyz + w\bar{x}yz + wx\bar{y}z + wxy\bar{z} \\ H(w,x,y,z) &= wx+wy+wz+xy+xz+yz \end{aligned}$$

Recall that, for any Boolean functions A, B of degree 4, one writes $A \leq B$ if and only if $\forall w,x,y,z \in \{0,1\}$, $A(w,x,y,z) \leq B(w,x,y,z)$. Also recall that, if $A \leq B$, then $A(w,x,y,z)\,B(w,x,y,z) = A(w,x,y,z)$ for all $w,x,y,z \in \{0,1\}$.

a) Prove or disprove each of the following: $F \leq G$, $F \leq H$, $G \leq F$, $G \leq H$, $H \leq F$ and $H \leq G$.

b) Design a circuit using inverters, AND and OR gates, that implements the Boolean function

$$L(w,x,y,z) = F(w,x,y,z)\,G(w,x,y,z)\,H(w,x,y,z).$$

You may, if you wish and if you provide an explanation, simplify this Boolean expression to obtain a simpler circuit producing the same output.

Exercise 2. A mailbox contains 1,000,000 e-mails, each one having from 1 word to 1,000 words. Altogether, the mails have a total of 54,000,000 words. In the mails, 50,000 distinct words can be found. One of these words is "hello."
Prove or disprove each of the statements below:

a) One mail is at least 55 words long.
b) At least a thousand of the mails contain the same number of words.
c) One word has a total number of occurences, in all mails, of 1000 or more.
d) Two mails, at least, contain both the same number of words and the same number of occurences of the word "hello."

Exercise 3. Solve the following recurrence relations:

a) $x_n = -\frac{3}{4}x_{n-1} - \frac{1}{8}x_{n-2}$, with initial conditions $x_0 = x_1 = 1$.
b) $y_n = \quad 2y_{n-1} - \frac{8}{9}y_{n-2}$, with initial conditions $y_0 = y_1 = 1$.

Exercise 4. Let $A = \{0,1\}^5$ be the set of binary sequences (strings) of zeros and ones of length 5. We will write $X = (x_1, x_2, \ldots, x_5)$ an element of A, where $x_i \in \{0,1\}$ for all $i \in \{1, \ldots, 5\}$. Also, e.g. $(1,0,1,0,0)$ will be written 10100.

a) Let f be the function defined by

$$f : \left\{ \begin{array}{lcl} A & \longrightarrow & \{0 \ldots 5\} \\ X & \longrightarrow & x_1 + \ldots + x_5, \end{array} \right.$$

and let R be the relation defined on A by

$$R(X,Y) \equiv f(X) = f(Y).$$

1) Prove or disprove that f is is onto.
2) Prove or disprove that f is one-to-one.
3) Prove that R is an equivalence relation.
4) Write the equivalence classes of 01010 and of 10000.
5) Write the cardinal of the equivalence class of X as a function of $f(X)$.

b) Let S be the relation defined on A by

$$S(X,Y) \equiv \forall i \in \{1, \ldots, 5\}, \, x_i \leq y_i.$$

1) Prove that S is a partial order relation.
2) Prove or disprove that S is a total order.
3) Write the set of elements $Y \in A$ such that $S(Y, 10100)$.
4) Write the set of upper and lower bounds of the set $\{10100, 01010\} \subseteq A$.
5) Write the sets of maximal and of minimal elements of A and, if they exist, the greatest and least element of A.