DISCRETE MATH - CS275 MIDTERM 2 - SOLUTION

NOVEMBER 16TH 2004, 12:30-13:45

Each of the 4 exercises is worth 5 points.
No calculator is needed or allowed in this exam.
All answers must come with an explanation.

Advice:

- Read all the exercises before answering to any.
- Spend 10 to 15 minutes on each exercise, then return to questions you missed.
- For each exercise, read **each word** with the greatest care and **without hurrying**. If needed, read it **many times**, until the meaning of the questions is clear.
- Check for extra information on the blackboard.

Exercise 1. Let F, G and H be Boolean functions of degree four, defined by

$$\begin{aligned} F(w,x,y,z) &= w+x+y+z \\ G(w,x,y,z) &= \bar{w}xyz + w\bar{x}yz + wx\bar{y}z + wxy\bar{z} \\ H(w,x,y,z) &= wx + wy + wz + xy + xz + yz \end{aligned}$$

Recall that, for any Boolean functions A, B of degree 4, one writes $A \leq B$ if and only if $\forall w,x,y,z \in \{0,1\}$, $A(w,x,y,z) \leq B(w,x,y,z)$. Also recall that, if $A \leq B$, then $A(w,x,y,z)\,B(w,x,y,z) = A(w,x,y,z)$ for all $w,x,y,z \in \{0,1\}$.

a) Prove or disprove each of the following: $F \leq G$, $F \leq H$, $G \leq F$, $G \leq H$, $H \leq F$ and $H \leq G$.
Solution:
1) $F \leq G$: False: $F(0,0,0,1) = 1$, $G(0,0,0,1) = 0$.
2) $F \leq H$: False: $F(0,0,0,1) = 1$, $H(0,0,0,1) = 0$
3) $G \leq F$: True. Seen from truth table.
4) $G \leq H$: True. Seen from truth table.
5) $H \leq F$: True. Seen from truth table.
6) $H \leq G$: False: $H(0,0,1,1) = 1$, $G(0,0,1,1) = 0$

b) Design a circuit using inverters, AND and OR gates, that implements the Boolean function

$$L(w,x,y,z) = F(w,x,y,z)\,G(w,x,y,z)\,H(w,x,y,z)\,.$$

You may, if you wish and if you provide an explanation, simplify this Boolean expression to obtain a simpler circuit producing the same output.
Solution:
Since $G \leq F$, one has $FG = G$. Since moreover $G \leq H$, one has $GH = G = (FG)\,H = FGH$. So one has $L = G$ and the circuit in Fig. implements L.
It seems everyone knows how to draw the circuit of a Boolean circuit (good!), so I do not include a figure here.

Exercise 2. A mailbox contains 1,000,000 e-mails, each one having from 1 word to 1,000 words. Altogether, the mails have a total of 54,000,000 words. In the mails, 50,000 distinct words can be found. One of these words is "hello."
Prove or disprove each of the statements below:

a) One mail is at least 55 words long.
Solution: No: Proof by counter-example. Consider a mailbox of 1,000,000 mails of length 54, so that 50e3 distinct words are present. The total number of words is 54e6 and no mail has 55 or more words.

b) At least a thousand of the mails contain the same number of words.
Solution: Yes: Proof using the pigeonheole principle. Let w_i, for $i \in \{1, \ldots, 1000\}$ be the number of mails containing i words. One has $w_1 + w_2 + \ldots + w_{1000} = 1,000,000$, so that, by the pigeonhole principle, $w_i \geq \lceil 1,000,000/1,000 \rceil = 1000$ for some i.

c) One word has a total number of occurences, in all mails, of 1000 or more.
Solution: Yes: use the pigeonhole principle. Let y_n ($n \in \{1, \ldots, 50,000\}$) be the total number of occurences of word n. Since $y_1 + \ldots + y_{50,000} = 54e6$, the pigeonhole principle says that the number of occurences y_n of one word at least is $\lceil 54e6/5e5 \rceil = 1,080$ or greater.

d) Two mails, at least, contain both the same number of words and the same number of occurences of the word "hello."
Solution: Yes. Proof using the pigeonheole principle. Let $p = (x, z)$ be, a pair of numbers, representing respectively the number of words and the number of occurences of "hello" in a message. One has $0 \leq z \leq x \leq 1000$ and also $0 < x$. The possible pairs are thus $p_1 = (1,0)$, $p_2 = (1,1)$, $p_3 = (2,0)$, $\ldots, p_{500,499} = (1000, 1000)$. Let u_i be the number of mails for which the number of words x and occurences of "hello" z are equal to p_i. Since the u_i sum up to the number of e-mails, i.e. $u_1 + \ldots + u_{500,499} = 1e6$, by the pigeonhole principle, one of the u_i must be greater or equal to $\lceil 1e6/500,499 \rceil = 2$.

Exercise 3. Solve the following recurrence relations:

a) $x_n = -\frac{3}{4}x_{n-1} - \frac{1}{8}x_{n-2}$, with initial conditions $x_0 = x_1 = 1$.
Solution: The characteristic polynomial $r^2 + \frac{3}{4}r + \frac{1}{8}$ has distinct roots $r_1 = -\frac{1}{4}$ and $r_2 = -\frac{1}{2}$. From the initial conditions, one gets

$$x_n = 6\left(-\frac{1}{4}\right)^n - 5\left(-\frac{1}{2}\right)^n$$

b) $y_n = \quad 2y_{n-1} - \frac{8}{9}y_{n-2}$, with initial conditions $y_0 = y_1 = 1$.
Solution: The characteristic polynomial $r^2 - 2r + \frac{8}{9}$ has distinct roots $r_1 = \frac{4}{3}$ and $r_2 = \frac{2}{3}$. From the initial conditions, one gets

$$y_n = \frac{1}{2}\left(\frac{4}{3}\right)^n + \frac{1}{2}\left(\frac{2}{3}\right)^n.$$

One may check that $y_2 = \frac{10}{9}$, $y_3 = \frac{12}{9} = \frac{4}{3}$.

Exercise 4. Let $A = \{0,1\}^5$ be the set of binary sequences (strings) of zeros and ones of length 5. We will write $X = (x_1, x_2, \ldots, x_5)$ an element of A, where $x_i \in \{0,1\}$ for all $i \in \{1, \ldots, 5\}$. Also, e.g. $(1,0,1,0,0)$ will be written 10100.

a) Let f be the function defined by

$$f : \begin{cases} A & \longrightarrow \quad \{0 \ldots 5\} \\ X & \longrightarrow \quad x_1 + \ldots + x_5, \end{cases}$$

and let R be the relation defined on A by

$$R(X, Y) \equiv f(X) = f(Y).$$

1) Prove or disprove that f is is onto.
Solution: Yes: For any i in the co-domain$\{0,\ldots,5\}$ of f, on has $f(\underbrace{1\ldots1}_{i}\underbrace{0\ldots0}_{5-i}) = i$, so that f is onto.

2) Prove or disprove that f is one-to-one.
Solution: No: $f(1,0,\ldots,0) = f(0,1,0,0,0) = 1$.

3) Prove that R is an equivalence relation.
Solution:
- R is (trivially) reflexive, since $\forall X,\ f(X) = f(X)$.
- R is (trivially) symmetric since for all X, Y, one has $f(X) = f(Y) \Longrightarrow f(Y) = f(X)$.
- R is (trivially) transitive since for all X, Y, Z, one has $f(X) = f(Y) \wedge F(Y) = f(Z) \Longrightarrow f(X) = f(Z)$.

4) Write the equivalence classes of 01010 and of 10000.
Solution:
- $\bar{X}_1 = \{X \in A \,|\, f(X) = 2\} = \{11000, 10100, 10010, 10001, 01100, 01010, 01001, 00110, 00101, 00011\}$.
- $\bar{X}_2 = \{X \in A \,|\, f(X) = 1\} = \{10000, 01000, 00100, 00010, 00001\}$.

5) Write the cardinal of the equivalence class of X as a function of $f(X)$.
Solution: $|\bar{X}| = C(5, f(X))$.

b) Let S be the relation defined on A by

$$S(X,Y) \equiv \forall i \in \{1,\ldots,5\},\ x_i \leq y_i.$$

1) Prove that S is a partial order relation.
Solution:
- S is (trivially) reflexive, since $\forall X,\ \forall i \in \{1,\ldots 5\}\, x_i \leq x_i$.
- R is (trivially) antisymmetric since, for all X, Y, if $\forall i \in \{1,\ldots 5\}\, x_i \leq y_i$ and $\forall i \in \{1,\ldots 5\}\, y_i \leq x_i$, then $\forall i,\ x_i = y_i$.
- R is (trivially) transitive since for all X, Y, Z, if $\forall i \in \{1,\ldots 5\}\, x_i \leq y_i$ and $\forall i \in \{1,\ldots 5\}\, y_i \leq z_i$, then $\forall i,\ x_i \leq z_i$.

2) Prove or disprove that S is a total order.
Solution: Let $Y_1 = (1,0\ldots0) \in A$ and $Y_1 = (0,1,0,\ldots,0) \in A$. Since $\neg S(Y_1,Y_2)$ and $\neg S(Y_2,Y_1)$, i.e. Y_1 and Y_2 are not comparable, S is not a total order.

3) Write the set of elements $Y \in A$ such that $S(Y, 10100)$.
Solution: $\{00000, 10000, 00100, 10100\}$.

4) Write the set of upper and lower bounds of the set $\{10100,\ 01010\} \subseteq A$.
Solution: Upper bounds: $\{11110, 11111\}$. There is a single lower bound: 00000.

5) Write the sets of maximal and of minimal elements of A and, if they exist, the greatest and least element of A.
Solution: It is clear that $(\underbrace{1,\ldots,1}_{n})$ is greater than all other elements and is thus the greatest element and the only maximal element; and that $(\underbrace{0,\ldots,0}_{n})$ is less than all other elements and is thus the least element and the only minimal element.