Three solved combinatorics problems

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Abstract

Here are three simple combinatorics problems and one or two ways to solve each.

1 Bi-color 4-hands with 2 cards of each color

In a deck, there are 32 cards. Each card is of one of four possible suits and one of eight possible kinds, so that the deck can be identified with the set

$$\mathcal{D} = \{(s,k) \mid s \in \{1,2,3,4\},\ k \in \{1,...,8\}\} = \{1,2,3,4\} \times \{1,...,8\}.$$

A hand of 4 cards is a set of 4 distinct cards of the deck, i.e. the order of the card does not matter.

a) Write in mathematical notation the set of 4-hands in which two cards of one suit and two cards of another?

b) What is the cardinal of this set?

1.1 Short solution

Question 1

For conciseness of notation, assume variables "s_i" belong to $\{1...4\}$ and "k_i" belong to $\{1...8\}$.

The set in question can be written

$$T = \left\{ \ \{(s_1,k_1),(s_1,k_2),(s_2,k_3),(s_2,k_4)\} \ \middle| \ s_1 < s_2,\ k_1 < k_2,\ k_3 < k_4 \ \right\}$$

Question 2

Any element in this set is defined by

a) a pair of distinct colors $\{s_1, s_2\} \subseteq \{1...4\}$,

b) two distinct kinds $\{k_1, k_2\} \subseteq \{1...8\}$ and

c) two distinct kinds $\{k_3, k_4\} \subseteq \{1...8\}$.

Each triplet $(\{s_1,s_2\}, \{k_1,k_2\}, \{k_3,k_4\})$, defines a 4-hand that belongs to T. Moreover, all hands in T can be defined by such a triplet and two distinct triplets define two distinct elements of T, so that, using the product rule, one gets

$$|T| = C(4,2) \cdot C(8,2) \cdot C(8,2) = 4704.$$

1.2 Detailed solution

Question 1

The set in question can be written

$$T = \left\{ \{(s_1,k_1),(s_1,k_2),(s_2,k_3),(s_2,k_4)\} \;\middle|\; \begin{array}{c} s_1,s_2 \in \{1...4\},\ \forall i\, k_i \in \{1...8\} \text{ and} \\ s_1 \neq s_2,\ k_1 \neq k_2,\ k_3 \neq k_4 \end{array} \right\}.$$

Question 2

For notational convenience, define $\wp_p(n) = \{A \mid A \subseteq \{1...n\}, |A| = p\}$ be the set of subsets of $\{1...n\}$ that have exactly p elements.

We now define a mapping $F : \wp_2(4) \times \wp_2(8) \times \wp_2(8) \longrightarrow T$. Note that:

- For any $S \in \wp_2(4)$, there exist a unique pair (s_1,s_2) s.t. $s_1 < s_2$ and $\{s_1,s_2\} = S$.
- For any $K \in \wp_2(8)$, there exist a unique pair (k_1,k_2) s.t. $k_1 < k_2$ and $\{k_1,k_2\} = K$.
- For any $L \in \wp_2(8)$, there exist a unique pair (k_3,k_4) s.t. $k_3 < k_4$ and $\{k_3,k_4\} = L$.

Any triplet $(S,K,L) \in \wp_2(4) \times \wp_2(8) \times \wp_2(8)$ thus uniquely defines

$$F(S,K,L) = \{(s_1,k_1),(s_1,k_2),(s_2,k_3),(s_2,k_4)\}$$

and it is clear that $F(S,K,L)$ belongs to T, for all S, K and L.

Moreover, any element of $H \in T$ defines two suits s_1, s_2 and two pairs of kinds $\{k_1,k_2\}$ and $\{k_3,k_4\}$, such that $F(\{s_1,s_2\},\{k_1,k_2\},\{k_3,k_4\}) = H$, so that F is *onto*.

Finally, it is clear that two distinct triplets of parameters $(S,K,L) \neq (S',K',L')$ will result in distinct hands $F(S,K,L) \neq F(S',K',L')$, so that F is *one-to-one*.

So, by the product rule, one has

$$|T| = |\wp_2(4)| \cdot |\wp_2(8)| \cdot |\wp_2(8)| = C(4,2) \cdot C(8,2) \cdot C(8,2) = 4704.$$

1.3 Comments

Even the "detailed" solution could be made more detailed. In particular, the facts that the mapping F is onto and one-to-one are more "said" than "proved." Why is that? and why is that acceptable?

- First, the arguments used to show the "onto" and "one-to-one" properties indicate how to proceed to a full proof.
- Second, the full proof would be longer and involve more notation. Overall, the explanation would be more confusing to the reader.

2 Strings made of letters and a single "@"

How many distinct strings of length n are there such that

a) There is a single "@" in the string and it is neither at the beginning nor at the end.

b) All other characters are in the range `[A-Z]`.

2.1 Intuition

Examples

Let´s look at[1] strings that match the definition above, with e.g. $n = 5$. "ABC@CD", "A@BCD", "ABC@D" are all ok. "@ABCD" and "ABCD@" are not, since the "@" should neither be at the beginning nor at the end of the string.

The natural way of writing a string[2] is to use a sequence: "AB@CD" $= (\mathtt{A}, \mathtt{B}, \mathtt{@}, \mathtt{C}, \mathtt{D})$, "A@BCD" $= (\mathtt{A}, \mathtt{@}, \mathtt{B}, \mathtt{C}, \mathtt{D})$ etc ... "ABCD@" $= (\mathtt{A}, \mathtt{B}, \mathtt{C}, \mathtt{D}, \mathtt{@})$.

Formalizing things We are given a counting problem in English. In mathematics, the question will be "what is the cardinal of this set." Before answering the question, we must find out what the set is. One suitable set is:

$$\{(c_1, ..., c_n) \mid \exists!\, i \in \{2, ..., n-1\},\ c_i = \texttt{"@"} \text{ and } \forall j \neq i,\ c_j \in \{\texttt{"A"}, ..\texttt{"Z"}\}\}\,. \tag{1}$$

The first condition ($\exists! i$..."@") says that there is a unique "@" in the string and that it is neither at the beginning nor at the end. The second condition ($\forall j \neq i$...) says that all other characters are upper-case letters.

Looking at the problem from another angle, one sees that another suitable set is

$$\begin{array}{ll} \overbrace{L \times \{\texttt{"@"}\} \times L \times L \times ... \times L}^{n} & \cup \\ L \times L \times \{\texttt{"@"}\} \times L \times ... \times L & \cup \\ \vdots & \\ L \times ... \times L \times \{\texttt{"@"}\} \times L \times L & \cup \\ L \times ... \times L \times L \times \{\texttt{"@"}\} \times L, & \end{array} \tag{2}$$

where $L = \{\texttt{"A"}, ..\texttt{"Z"}\}$. This set is longer to write. However,

a) it is clear that the terms in the union are two-by-two disjoint, so that the sum rule[3] applies. And also,

b) the product rule tells us how to compute the cardinal of each term in the union.

So which seems the best? Since the answer to the problem seems just a few steps away from Eq. 2, let´s go for that option.

2.2 Solution (v.1)

Let $L = \{\texttt{"A"}, ..\texttt{"Z"}\}$. Define

$$S = \begin{array}{ll} \overbrace{L \times \{\texttt{"@"}\} \times L \times L \times ... \times L}^{n} & \cup \\ L \times L \times \{\texttt{"@"}\} \times L \times ... \times L & \cup \\ \vdots & \\ L \times ... \times L \times \{\texttt{"@"}\} \times L \times L & \cup \\ L \times ... \times L \times L \times \{\texttt{"@"}\} \times L. & \end{array}$$

It is clear that this set contains all strings of length n that verify conditions **a)** and **b)** above. Moreover, the $n-2$ terms of the union are two-by-two disjoint, so that the sum

[1] As always, it is good practice to look at what objects are being asked for.
[2] Since the order of the letters in a string matters.
[3] See Section 4.1 of the textbook [1].

rule applies and the cardinal of each term, by the product rule is $1 \cdot |L| \cdot \ldots \cdot |L| = |L|^{n-1}$. In all, there are thus

$$(n-2) \cdot 26^{n-1}$$

distinct elements in S.

2.3 Solution (v.2)

Comment

The above solution is the simplest. Let´s suppose that we did not think of using a union of disjoint Cartesian products to represent the set of strings and that we only came out w/ Eq. (1). Let´s then use that equation to count the strings.

As seen in class, one way of counting the elements of a set is to find a bijective mapping from a set with known cardinal to that set.

The solution proper

Let $L = \{\texttt{"A"}, ..\texttt{"Z"}\}$ and write

$$S = \{(c_1, ..., c_n) \mid \exists!\, i \in \{2, ..., n-1\}\,,\, c_i = \texttt{"@"} \text{ and } \forall j \neq i,\, c_j \in \{\texttt{"A"}, ..\texttt{"Z"}\}\}$$

the set of strings verifying conditions **a)** and **b)**. Define

$$f\,:\,(i, a_1, \ldots, a_{n-1}) \in \{2..n-1\} \times L^{n-1} \longrightarrow (a_1, \ldots, a_{i-1}, \texttt{"@"}, a_i, \ldots, a_{n-1}) \in S.$$

It is clear that, indeed, $f\,(i, a_1, \ldots, a_{n-1}) \in S$ for all $(i, a_1, \ldots, a_{n-1}) \in \{2..n-1\} \times L^{n-1}$.

Let´s show that f is onto and one-to-one.

Let $(c_1, ..., c_n) \in S$ and let i be the unique index such that $c_i = \texttt{"@"}$. It is easy to see that $f\,(i, c_1, ..., c_{i-1}, c_{i+1}, ..., c_n) = (c_1, ..., c_n)$, so that f is onto.

Let $(i, a_1, \ldots, a_{n-1})$ and $\left(i', a'_1, \ldots, a'_{n-1}\right)$ be distinct elements of $\{2..n-1\} \times L^{n-1}$. It is easy to verify that, in that case, $f\,(i, a_1, \ldots, a_{n-1}) \neq f\,\left(i', a'_1, \ldots, a'_{n-1}\right)$, so that f is one-to-one.

Since f is both onto and one-to-one, the cardinal of its domain is equal to the cardinal of its co-domain[4] S, and thus

$$\begin{aligned} |S| &= \left|\{2..n-1\} \times L^{n-1}\right| \\ &= |\{2..n-1\}| \times |L|^{n-1} \\ &= (n-2) \cdot 26^{n-1}. \end{aligned}$$

More comments

We saw in the "Specimen of simple proofs" `http://www.vis.uky.edu/~etienne/cs275/helpers.html` that a solution to a problem may consist of some "notation," some "announcing of what will be done," some "transformation of statements" and some "justification." I recommend finding yourself the function of each sentence in the solutions given above.

Also, the "intuition" part above plays an important role, even though it is not necessary to put it in the solution.

[4]The words "onto", "one-to-one", "domain" and "co-domain" were defined in class and are also defined in Section 1.8 of the textbook [1].

3 Choosing 10 bagels among 8 different kinds

A shop sells 8 different types of bagels and has plenty (more than 10) of each type. You go buy 10 bagels and put them in a bag.

Question:

> How many different outcomes can your bagel-shopping trip have? All that matters is how many bagels of each type you bought.

3.1 Intuition

Comment

Even if it wasn´t said "all that matters is how many bagels of each type you bought," you should have been able to guess that part, since you put the bagels in a bag and all that *could* matter is how many bagels of each type you have.

Now, let´s see how to answer the question.

Think about the problem

The exposition of the exercise says there are 8 types of bagels, lets call them $\{1, ..., 8\}$.

What are possible outcomes of my shopping? For example, I can pick 5 bagels of type 1, 5 of type 2 and none of the other types. If I write one after the other the numbers of bagels of each kind, I get

$$(5, 5, 0, 0, 0, 0, 0, 0),$$

where the 1st and 2nd numbers are fives and all the other are zeros.

If instead I picked 1 bagel of types 1-7 and 3 bagels of type 8, I would get the sequence

$$(1, 1, 1, 1, 1, 1, 1, 3).$$

In general, the outcome my shopping would thus be a sequence

$$(x_1, x_2, ..., x_8),$$

where

- each x_i is a natural number (I am not buying half-bagels), so $\forall i \in \{1..8\}, x_i \in \mathbb{N}$. And moreover,
- the total number of bagels is 10, so that $x_1 + \ldots + x_8 = 10$.

Now, that begins to sound familiar and I can start writing the solution

3.2 Solution

The set of possible outcomes to my bagel-shopping trip is

$$\{(x_1, ..., x_8) \mid \forall i,\, x_i \in \mathbb{N} \text{ and } x_1 + \ldots + x_8 = 10\},$$

where x_i represents the number of bagels of type i that I bought. As seen in class[5], the cardinal of this set is

$$C(10 + 8 - 1, 8 - 1) = C(17, 7) = 19,448.$$

More comments

The solution is so short because the problem is almost a direct application of a result shown in class. You should recall that the full statement of the result[6] used above is :

For all natural numbers p and q,

$$|\{(x_1, ..., x_p) \mid \forall i,\, x_i \in \mathbb{N},\, x_1 + \ldots + x_p = q\}| = C(p + q - 1, p - 1).$$

[5] See also Theorem 2 in Section 4.5 of [1].

[6] The textbook [1] writes $C(p + q - 1, q)$ in the right-hand side of the equation, in theorem 2 of Section 4.5.

3.3 Variant : at least one bagel of each type

A shop sells 8 different types of bagels and has plenty (more than 10) of each type. You go buy 10 bagels, at least one of each type, and put them in a bag. How many different outcomes can your bagel-shopping trip have?

3.4 Intuition

This is the same problem as above, except that each x_i is 1 or more. So, the set of possible outcomes is

$$\{(x_1, ..., x_8) \mid \forall i,\, x_i \in \mathbb{N} \text{ and } x_i \geq 1 \text{ and } x_1 + \ldots + x_8 = 10\}.$$

At that point, either you remember that the cardinal of this type of set was given during the class and you write immediately the answer (w/ enough justification, of course), or you don´t remember that result and you keep on thinking...

a) Note that, if each x_i is 1 or greater, then $x_i - 1$ is 0 or greater. Let´s define $y_i = x_i - 1$.

b) What is the sum of the y_i? $y_1 + \ldots + y_8 = x_1 - 1 + \ldots + x_8 - 1 = 10 - 8 = 2$.

c) For any outcome $(x_1, \ldots, x_8) \in \mathbb{N}^8$ s.t. $\forall i,\, x_i \geq 1$ and $x_1 + \ldots + x_8 = 10$, I can define a $(y_1, \ldots, y_8) \in \mathbb{N}^8$ s.t. $y_i = x_i - 1$, $y_i \geq 0$ and $y_1 + \ldots + y_8 = 2$.

d) Moreover, I can "put the outcomes $(x_1, ..., x_8)$ in one-to-one correspondence w/ the set of $(y_1, ..., y_8)$," and I know (from class) that the number of elements of the set

$$\{(y_1, ..., y_8) \mid \forall i,\, y_i \in \mathbb{N} \text{ and } y_1 + \ldots + y_8 = 2\}$$

is

$$C\,(2 + 8 - 1, 8).$$

So I can write the solution in the following way:

3.5 Solution

The set of possible outcomes to my bagel-shopping trip is

$$O = \{(x_1, ..., x_8) \mid \forall i,\, x_i \in \mathbb{N} \text{ and } x_i \geq 1 \text{ and } x_1 + \ldots + x_8 = 10\},$$

where x_i is the number of bagels of type i that I bought. By defining $y_i = x_i - 1$ and noting that $\forall i,\, y_i \geq 0$ and that $y_1 + \ldots + y_8 = 2$, one sees that the set of outcomes is mapped to the set

$$P = \{(y_1, ..., y_8) \mid \forall i,\, y_i \in \mathbb{N} \text{ and } y_1 + \ldots + y_8 = 2\}.$$

Also, one easily shows that

a) the mapping is onto, i.e. all $(y_1, \ldots, y_8)$ in P can be defined from a $(x_1, \ldots, x_8)$ in O. And moreover,

b) The mapping is one-to-one, since distinct outcomes $(x_1, \ldots, x_8)$ and $(x'_1, \ldots, x'_8)$ yield distinct sequences $(y_1, \ldots, y_8)$ and $(y'_1, \ldots, y'_8)$.

So the number of possible outcomes to my bagel-shopping trip is the cardinal of P, which is, as seen in class

$$C\,(2 + 8 - 1, 8 - 1) = 36.$$

References

[1] K. H. Rosen. *Discrete Mathematics and Its Applications.* Mc Graw Hill, 5 edition, 2003.