English-to-math phrase book

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This document shows how to translate some statements from English into the language of mathematics. The detail of these translations is voluntarily exaggerated.

Contents

1 Statements

1.1 All natural numbers are greater than or equal to zero

This statement means the same thing as (use the singular)

> Any natural number is greater than or equal to zero

which is equivalent to (give the name n to the number)

> Any natural number n is greater than or equal to zero

which is equivalent to (use $n \in \mathbb{N}$ instead of "natural number n")

> Any $n \in \mathbb{N}$ is greater than or equal to zero

which is equivalent to (use "for all" instead of "any")

> For all $n \in \mathbb{N}$, n is greater than or equal to zero

which is equivalent to (use $\forall$ instead of "for all" and use the $\geq$ sign)

$$\forall n \in \mathbb{N},\ n \geq 0.$$

Variant, using the implication sign $\Longrightarrow$

The above statement

> Any natural number n is greater than or equal to zero.

is equivalent to

> Any object n that is a natural number is greater than or equal to zero.

which is equivalent to (add "given" and use a "if ... then ..." form)

> Given any object n, if n is a natural number, then n is greater than or equal to zero.

which is equivalent to (replace "is natural number" by $\in \mathbb{N}$ and use $\geq$)

> Given any object n, if $n \in \mathbb{N}$, then $n \geq 0$

which is equivalent to (use $\Longrightarrow$ instead of "if ... then ..." and add parentheses)

> Given any object n, $(n \in \mathbb{N} \Longrightarrow n \geq 0)$

which is equivalent to (replace "given any object" by $\forall$)

$$\forall n\ (n \in \mathbb{N} \Longrightarrow n \geq 0)\,.$$

1.2 The empty set contains no element

This statement is equivalent to

> No element is contained in the empty set

which is equivalent to (introducing a variable name)

> For all element x, x does not belong to the empty set

which is equivalent to (writing the empty set as $\emptyset$)

> For all element x, x does not belong to $\emptyset$

which is equivalent to (writing "does not belong" as $\notin$)

> For all element x, $x \notin \emptyset$

which is equivalent to (writing "for all element" as $\forall$)

$$\forall x,\ x \notin \emptyset. \tag{1}$$

FYI: this is the "empty set axiom" of set theory.

A specific case: 3 does not belong to the empty set

This statement means the same thing as the statement (use the symbol $\emptyset$)

> 3 does not belong to $\emptyset$

and then

> $3 \notin \emptyset$.

$\boxed{\textbf{In this section, let } x = \{3, 5\}}$

1.3 The set x contains the numbers 3 and 5

This statement is equivalent to

> 3 belongs to x and 5 belongs to x

which is equivalent to (use $\in$ for "belongs to")

$$3 \in x \,\text{and}\, 5 \in x$$

or, using $\wedge$ for "and"

$$3 \in x\ \wedge 5 \in x.$$

In this section, let $x = \{3, 5\}$

1.4 The set x contains only the numbers 3 and 5

This statement is equivalent to

> Any element contained in x is either 3 or 5

which is equivalent to saying that (give the name y to the element in x)

> Any element y contained in x is either 3 or 5

is equivalent to saying that (slight rephrasing)

> Any y contained in x is either equal to 3 or to 5

is equivalent to saying (replace "any y contained in x" by $\forall y \in x$

> $\forall y \in x$, y is either equal to 3 or to 5

is equivalent to saying (use $=$ for the equalities and add parentheses)

$$\forall y \in x,\ (y = 3 \,\text{or}\, y = 5)\,. \tag{2}$$

Variant using an implication $\Longrightarrow$

> Any element y contained in x is either 3 or 5

is equivalent to saying that (use the sign $=$ for "equal")

> Any element y contained in x either verifies $y = 3$ or verifies $y = 5$

is equivalent to saying that (use a "if ... then" expression)

> For any y, if y is contained in x, then it verifies either $y = 3$ or $y = 5$

is equivalent to saying that (use $\in$ for "is contained")

> For any y, if $y \in x$, then it verifies either $y = 3$ or $y = 5$

is equivalent to saying that ("it verifies either" is not useful any more)

> For any y, if $y \in x$, then $y = 3$ or $y = 5$

is equivalent to saying that (use $\forall$ in place of "for all" and $\Longrightarrow$ to represent the implication; add parentheses)

$$\forall y,\ y \in x \implies (y = 3 \quad \text{or} \quad y = 5)\,. \tag{3}$$

Note: The statements (2) and (3) are both true statements about x. These two statements are equivalent to the preferred notation

$$x = \{3, 5\}\,. \tag{4}$$

1.5 Given two objects, there is a set that contains just these two objects

This statement is equivalent to saying that (name the objects)

> Given any two objects x and y, there is a set z that contains just these two objects

which is equivalent to (slight rephrase)

> Given any x, given any y, there is a set z such that z contains just these two objects

which is equivalent to (replace "given any two objects" by $\forall$ symbols and "there is" by $\exists$)

> $\forall x\, \forall y\, \exists z$ such that z contains just these two objects.

There are (at least) two possibilities to proceed from here:

Possibility 1

Use the curly brace notation $\{x, y\}$ to represent the set z, yielding

> $\forall x\, \forall y\, \exists z$ such that $z = \{x, y\}$

and then (removing the now useless "such that")

$$\forall x\, \forall y\, \exists z,\ z = \{x, y\} \tag{5}$$

Possibility 2

Express "z contains just x and y" as in Sec. 1.4

> $\forall x\, \forall y\, \exists z$ such that any element contained in z is either x or y

which is the same as (give the name a to the "any element")

> $\forall x\, \forall y\, \exists z$ such that any element a contained in z is either x or y

which is the same as (proceed as in Sec. 1.4)[1]

[1] Note that this is the same as Eq. (2) except that 3 is replaced by x, 5 is replaced by y and x is replaced by z.

$$\forall x \,\forall y \,\exists z \text{ such that } \forall a \in z,\ (a = x \quad \text{or } a = y)$$

which is the same as (remove the useless "such that")

$$\forall x \,\forall y \,\exists z \,\forall a \in z,\ (a = x \quad \text{or} \ \ a = y)\,. \tag{6}$$

As in Sec. 1.4, one can use the variant with the $\Longrightarrow$:

$$\forall x \,\forall y \,\exists z \,\forall a,\ a \in z \Longrightarrow (a = x \quad \text{or} \ \ a = y)\,. \tag{7}$$

FYI: this last statement is the "pair axiom" of set theory.

1.6 Exercise: Two sets x and y are equal if and only if x is included in y and y is included in x

Translate this statement into a mathematical expression by following steps similar to those above.

1.7 Exercise: Given any natural number n, there exists a number greater than n

Translate this statement into a mathematical expression by following steps similar to those above.

2 Objects

2.1 The set of even numbers

is (giving the name m to the number)

the set of numbers m that are even

is (using the "set builder" notation)

$\{m \in \mathbb{N} \mid m \text{ is even}\}$

is (by definition of an even number)

$\{m \in \mathbb{N} \mid m \text{ is a multiple of two}\}$

is (by definition of a multiple of two)

$\{m \in \mathbb{N} \mid m \text{ is two times another number}\}$

is (naming n the "other number")

$\{m \in \mathbb{N} \mid m \text{ is two times another number } n\}$

is (adding "for some n" explicitly)

$\{m \in \mathbb{N} \mid m \text{ is two times another number } n, \text{ for some } n\}$

is (math notation for equality)

$\{m \in \mathbb{N} \mid m = 2n, \text{ for some } n\}$

is (putting "for some n" in front)

$\{m \in \mathbb{N} \mid \text{for some } n,\ m = 2n\}$

is (using $\exists$ instead of "for some")

$$\{m \in \mathbb{N} \mid \exists n,\ m = 2n\}\,. \tag{8}$$

The set of even numbers (bis)

is (by definition of an even number)

The set of numbers that are multiples of two

is (by definition of a multiple of two)

The set of numbers that are equal to two times another number

is (giving names m and n to the numbers involved)

The set of numbers m that are equal to two times another number n

is (slight rephrasing)

The set of numbers m such that m is two times another number n

is (adding "for some n" explicitly)

The set of numbers m such that m is two times another number n, for some n

is (math notation for equality)

The set of numbers m such that $m = 2n$, for some n

is (putting "for some n" in front)

The set of numbers m such that, for some n, $m = 2n$

is (using $\exists$ instead of "for some")

The set of numbers m such that,$\exists n$, $m = 2n$

is (using the "set builder" notation)

$$\{m \in \mathbb{N} \mid \exists n, m = 2n\}.$$

2.2 The set of subsets of a set a

also known as

The *power set* of a

is (by definition of a subset)

The set of sets that are included in a

is (by give a name to the subset)

The set of sets x that are included in a

is (slight rephrasing)

The set of sets x such that x is included in a

is (using the $\subseteq$ symbol for inclusion)

The set of sets x such that $x \subseteq a$

is (using the bracket notation)

$$\{x \mid x \subseteq a\}$$

is (in common mathematical notation)

$$\wp(a)$$

2.3 Exercise: the set of odd numbers

Define the set of odd numbers and write an expression similar to Eq. (8) above to represent it.

Hint: A number m is odd if and only if it can be written as $2n + 1$ for some $n \in \mathbb{N}$. An alternative definition is that a number m is odd if and only if it is not even.

2.4 Exercise: the set of multiples of a number n

Define the set of multiples of a number n and write an expression similar to Eq. (8) above to represent it.

2.5 Exercise: the set of divisors of a number n

Definition

> A number m is a divisor of n if and only if there exists a number p such that $mp = n$.

One also say that "m divides n" or that "n is a multiple of m".

Define the set of divisors of a number n and write an expression similar to Eq. (8) above to represent it.

3 Basic set operations

3.1 Union of two sets

Notation

The union of two sets x and y is a set that is written

$$x \cup y$$

Definition of the union

> The union of x and y is the set of objects that belong to x **or** to y (or both).

This definition is equivalent to (use the notation $x \cup y$)

> $x \cup y$ is the set of objects that belong to x or to y (or both)

which is equivalent to (give name a to "the objects")

> $x \cup y$ is the set of objects a that belong to x or to y (or both)

which is equivalent to (remove the "or both", *which is always assumed to accompany* the "or" operator)

> $x \cup y$ is the set of objects a that belong to x or to y

which is equivalent to (rephrase)

> $x \cup y$ is the set of objects a, such that a belongs to x or a belongs to y

which is equivalent to (use $\in$ for "belongs")

> $x \cup y$ is the set of objects a, such that $a \in x \quad \text{or} \quad a \in y$

which is equivalent to (use curly braces to define the set)

> $x \cup y$ is the $\{a \mid a \in x \quad \text{or} \quad a \in y\}$

which is equivalent to (use $=$ instead of "is the")

$$x \cup y = \{a \mid a \in x \quad \text{or} \quad a \in y\} \,. \tag{9}$$

Belonging to the union as a logical condition

> An object belongs to the union of x and y if and only if it belongs to x **or** it belongs to y (or both).

This definition is equivalent to (use the notation $x \cup y$, give the name a to "the object" and remove the "or both", *which is always assumed to accompany* the "or" operator)

> An object a belongs to $x \cup y$ if and only if a belongs to x or it belongs to y.

which is equivalent to (rephrase the beginning)

> Given any object a, a belongs to $x \cup y$ if and only if a belongs to x or a belongs to y (or both)

which is equivalent to (use $\in$ for "belongs to")

> Given any object a, $a \in x \cup y$ if and only if $a \in x$ or $a \in y$

which is equivalent to (use the connector $\iff$ instead of "if and only if" and add parentheses)

> Given any object a, $a \in x \cup y \iff (a \in x \quad \text{or} \quad a \in y)$

which is equivalent to (use $\forall$ instead of "given any object")

$$\forall a,\, a \in x \cup y \iff (a \in x \quad \text{or} \quad a \in y)\,. \tag{10}$$

3.2 Intersection of two sets

Notation

The intersection of two sets x and y is a set that is written

$$x \cap y$$

Definition of the intersection

> The intersection of x and y is the set of objects that belong to x **and** to y.

This definition differs from that of the union of two sets only by the "and" that replaces the "or". By doing the same development as above, one finds that the intersection of two sets can be defined by:

$$x \cap y = \{a \mid a \in x \quad \text{and} \quad a \in y\}\,.$$

One also has the condition

An object belongs to the intersection of two sets x and y if and only if it belongs to x **and** it belongs to y

which is written, in mathematical language

$$\forall a,\ a \in x \cap y \iff (a \in x \quad \text{and} \quad a \in y)\,.$$

3.3 Complement of a subset x of a set y

This is also called

the complement of x in y.

Notation

The complement of a subset x of a set y can be written in many ways:

$$y \setminus x \quad \text{or} \quad \complement_y x \quad \text{or} \quad \bar{x}^y.$$

Often, there is no doubt about what the set y is, in which case one writes

$$\complement x \quad \text{or} \quad \bar{x},$$

and one says simply "the complement of x".

Definition

The complement of x in y is the set of elements of y that do not belong to x.

By proceeding as in Section 3.1, one obtains the equivalent definition

$$y \setminus x = \{a \mid a \in y \wedge a \notin x\} = \{a \in y \,|\, a \notin x\}$$

as well as the condition

$$a \in y \setminus x \iff (a \in y \quad \text{and} \quad a \notin x)\,.$$

3.4 Exercise: symmetric difference of two sets x and y

Notation

The symmetric difference of two sets can be written

$$x \nabla y \quad \text{or} \quad x \Delta y \quad \text{or} \quad x \ominus y$$

Definition

The symmetric difference of x and y is the set of elements that belong to x or to y but not to both.

Exercise

Translate the above definition of the symmetric difference of x and y into an expression of the same form as Eq. (9).

Note: There are many solutions. The symmetric may also be defined using the union, intersection and complement.