Non-Graded Set Theory Exercises

Solve by August 30th, 2004. Answers given during recitation.

Solve the following exercises from Rosen [1] Sec. 1.6:
1, 2, 3, 5, 6, 7, 8, 9, 27, 28, 15, 16, 23, 22. Try to solve exercise 29.

Note on exercise 1.

We have not defined real numbers in class, but that should not prevent you from solving this exercise.

Extra question: what axiom given in class is used to define these sets.

Notes on exercise 2.

1. Sets built using the "*set builder notation*" [1, p. 79] are those built using the "*axiom of separation*" given in class.

2. Letters can be represented by numbers (e.g. as in the ASCII code, or by numbers 1..26).

Notes on exercise 7.

1. Point **b)** In class, the integer 0 was represented by the empty set $\emptyset$, so that $\emptyset \in \{0\}$ is the same as $\emptyset \in \{\emptyset\}$, and the answer is "true". The textbook [1], on the other hand, does not represent numbers by sets[1], so that the correct answer is "False".
   In all cases in which the correct answer according to the book and that according to the class differs, **both answers will be counted as correct**.

2. Reminder on strict and non-strict inclusion: $x \subset y$ if and only if $x \subseteq y$ and $x \neq y$.

Exercise 16.

- $A = B \Longrightarrow \wp(A) = \wp(B)$: Trivial.

[1] In the textbook, numbers are "given", while I presented a way of defining them from sets.

- $$\begin{aligned}\wp(A)=\wp(B) &\iff \forall y,\ (y\in\wp(A) \iff y\in\wp(B))\\ &\implies \forall\{x\},\ (\{x\}\in\wp(A) \iff \{x\}\in\wp(B))\\ &\iff \forall x,\ (x\in A \iff x\in B)\\ &\iff A=B.\end{aligned}$$

  1. The first and fourth lines use just the axiom of extensionality.
  2. The second line replaces "any subset y of A" by "any single-element subset $\{x\}$ of A".
  3. The third line holds because $\{x\}=\{z\}$ is equivalent to $x=z$.

Note on exercise 17.

a) Trap: what if $a=b$? In that case $\{a,b,\{a,b\}\}=\{a,a,\{a,a\}\}=\{a,\{a\}\}$ contains just two elements.

Solving exercise 29.

An ordered pair could be any mathematical object (a set) which uniquely defines two elements that can be distinguished, one as being the first and the other the second. The set $\{\{a\},\{a,b\}\}$ allows to do this.

Step 1: existence

Given any two sets a, b the pair axiom allows us to define the sets $\{a\}$, $\{a,b\}$ and $\{\{a\},\{a,b\}\}$.

Step 2: define a pair

Define a single-variable predicate $P(x)$ that is true if and only if x is a set of the form $\{\{a\},\{a,b\}\}$ for some a,b. Define:

$$P(x) \overset{\Delta}{\iff} \exists a\,\exists b,\ x=\{\{a\},\{a,b\}\}. \tag{1}$$

The $\overset{\Delta}{\iff}$ symbol says that the left expression is defined to mean the same as the right expression, a little bit like the `#define` macro in C.

The right equality is a valid statement of set theory. It can be written more extensively as :

$$\forall y,\ y\in x \iff (y=\{a\}\wedge y=\{a,b\}).$$

In turn, $y=\{a\}$ is equivalent to $\forall z,\ z\in y \iff z=a$ and $y=\{a,b\}$ is equivalent to $\forall z,\ z\in y \iff (z=a\vee z=b)$. Altogether, Eq. (1) expands to:

$$P(x) \iff \left(\exists a\,\exists b,\ \forall y,\ y\in x \iff \begin{pmatrix}(\forall z,\ z\in y \iff z=a)\vee\\ (\forall z,\ z\in y \iff (z=a\vee z=b))\end{pmatrix}\right).$$

The shorthand notation Eq. (1) is much preferable.

Step 3: define the "first element"

Define a two-variable predicate $F(x, a)$ that is true if and only if a is the first element in the pair x. Define:

$$F(x, a) \overset{\Delta}{\iff} P(x) \wedge \{a\} \in x.$$

Again, the shorthand notation $\{a\} \in x$ can be expanded to

$$\exists y,\, y \in x \wedge (\forall z\, (z \in y \iff z = a)).$$

Step 4: define the "second element"

Define a two-variable predicate $S(x, b)$ that is true if and only if b is the second element in a pair. Define:

$$S(x, b) \overset{\Delta}{\iff} P(x) \wedge ((\exists a,\, a \neq b \wedge \{a, b\} \in x) \vee (x = \{\{b\}\})). \tag{2}$$

This definition can be expanded as above. Note that one cannot use the definition

$$S_2(x, b) \overset{\Delta}{\iff} P(x) \wedge (\exists a,\, \{a, b\} \in x),$$

because if $x = \{\{a\}, \{a, b\}\}$, then one always has $S_2(x, a)$ in addition to $S_2(x, b)$. Also, the definition

$$S_3(x, b) \overset{\Delta}{\iff} P(x) \wedge (\exists a,\, a \neq b \wedge \{a, b\} \in x),$$

falls short of the objective: one cannot represent an ordered pair whose first and second elements are equal, $x = \{\{a\}, \{a, a\}\} = \{\{a\}\}$. In that case, $S_3(x, a)$ would be a false statement. So one adds the alternative condition $(x = \{\{b\}\})$ to obtain Eq. (2).

We have shown that, for any two sets a and b, the set $\{\{a\}, \{a, b\}\}$ exists and it is possible to define it´s first and second elements unambiguously.

From now on we will use the shorthand notation $(a, b) \triangleq \{\{a\}, \{a, b\}\}$, where the $\triangleq$ symbol means that the notation at the left is defined to represent the object (set) at the left.

References

[1] K. H. Rosen. *Discrete Mathematics and Its Applications.* Mc Graw Hill, 5 edition, 2003.