Specimen of simple proofs

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Abstract

This document contains templates solutions of exercises of the form "show that ... ." The specimen exercises are somewhat like those in homeworks and exams. The solution is broken down to pieces and each piece is commented. This document does not discuss the mathematical intuition used to solve the exercises, but only **the form** given to the solution.

The solution you give to an exercise should be written in such a way that a reader, who would have read the question, understands your answer and how you came to it.

- Do you say what you are doing, so that the reader knows what is going on?
- Do you justify each step that needs justification?

Contents

1 "There exist" proof: proof by example

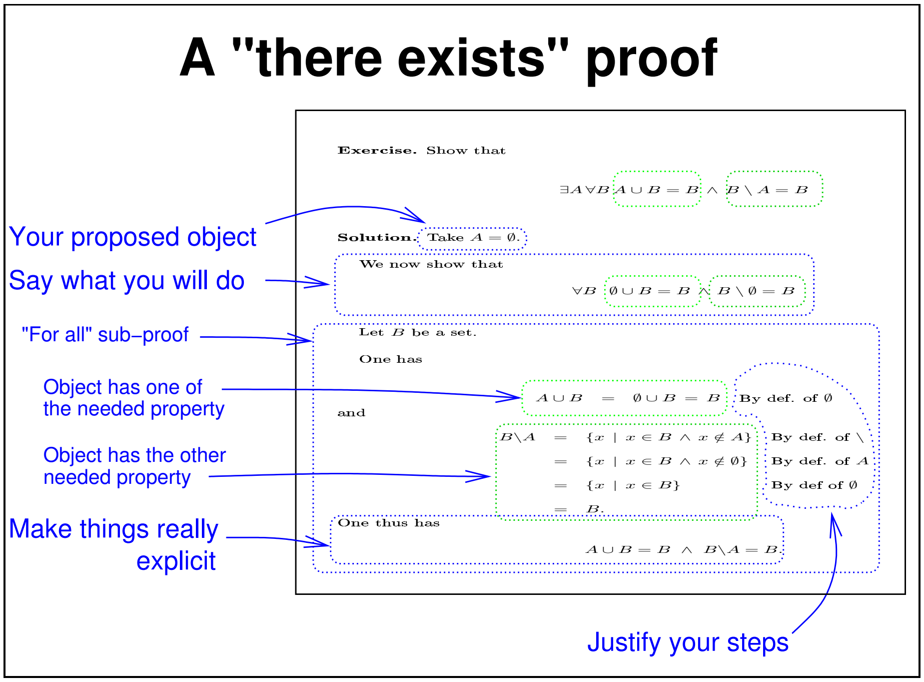

Figure 1: Proof by example.

Figure 1 shows a sample exercise and solution. The question is:

> Show that there exists a set A such that, for all set B, $A \cup B = B$ and $B \setminus A = B$,

where $B \setminus A$ is the set B minus A.

1.1 The solution

Since the question is to show that "there exists" something, it is enough to show one object (called A) that verifies the desired property (in green in Fig. 1).

Step 1: Choose an object

So the first step is to say what object you will show. This is the

> Take $A = \emptyset$.

part.

Step 2: Show that the object verifies the desired property

This is the sub-proof in Fig. 1. The property that A should verify is:

For all B, $A \cup B = B$ and $B \setminus A = B$

and I say that I will now prove it ("**say what you will do**", in Fig. 1), so that the reader knows what will come next.

If this property is to hold "for all B", I should take "any B" and show that the properties $A \cup B = B$ and $B \setminus A = B$ are true for it.

So I say

Let B be a set.

to say that B is "any set".

Step 2.1: Show that one of the properties is true for any B

The first property is very simple and I just say

$$A \cup B = \emptyset \cup B = B \qquad \text{(by def. of } \emptyset\text{)}$$

Saying "by def. of $\emptyset$" is not really needed, but it doesn't harm.

Step 2.2: Show that the other property is true for any B

The second property is simple too. You could say directly that

$$B \setminus \emptyset = B$$

In Fig. 1, there are lots of very detailed steps that could have been dispensed with. But again, too much detail does not usually harm.

Step 2.3: Put things together

This is the "**Make things really explicit**" step in the figure.

1.2 Variant

Exercise. Show that there exists a set A such that, for all set B, one has

$$A \cup B = B \wedge B \setminus A = B$$

Figure 2: An exercise exactly equivalent to that of Fig. 1.

2 "For all" proof

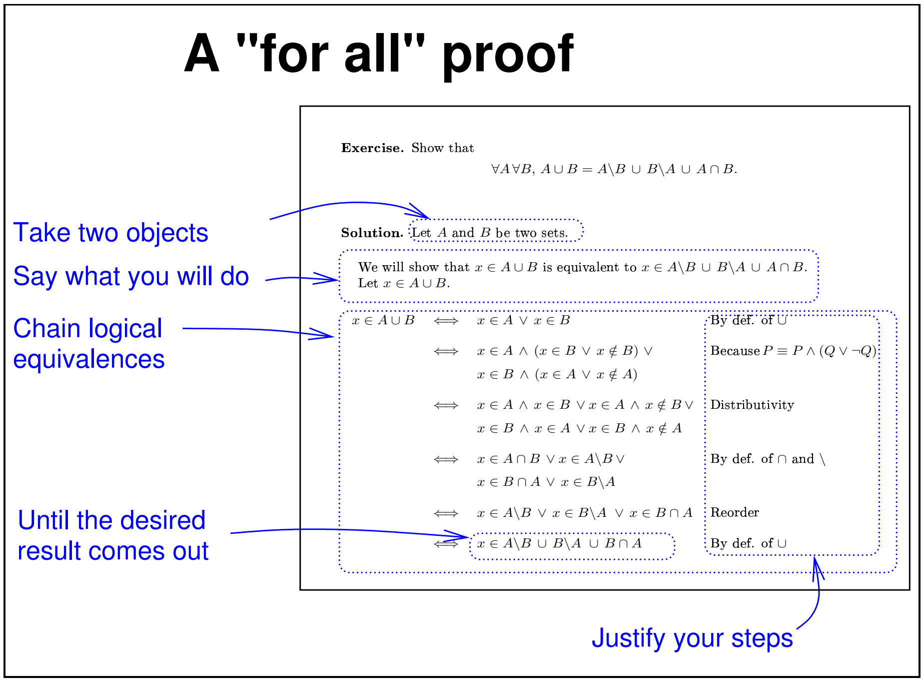

Figure 3: A proof by example.

Figure 3 shows a sample exercise and solution. The question is:

> Show that for all sets A and B, $A \cup B = A \backslash B \ \cup\ B \backslash A \ \cup\ A \cap B$.

2.1 The solution

Since the question is to show that "for all A and B", I should take any two sets and show that the desired property holds for them.

Step 1: Declare the object you will use

So the first step is to take two sets, which is done by saying

> Let A and B be two sets.[1]

Step 2: Show that these objects verify the desired property

Since the desired property

[1] If the question was about natural numbers, I would have said "let A, $B \in \mathbb{N}$."

$$A \cup B = A\backslash B \ \cup \ B\backslash A \ \cup \ A \cap B$$

is an equality of sets[2], I may prove it by showing that, for any element x, x belongs to one set if and only if it belongs to the other (this is the definition of set equality as I gave it in the first lecture).

So I **say what I will do**, to make sure the reader knows what to expect. Since I take an object x belonging to $A \cup B$, I say:

Let $x \in A \cup B$

Step 2.1: Show that $x \in A \cup B$ is equivalent to $x \in A\backslash B \ \cup \ B\backslash A \ \cup \ A \cap B$

This is done directly, by chaining logical equivalences, starting from $x \in A \cup B$ until $x \in A\backslash B \ \cup \ B\backslash A \ \cup \ A \cap B$ comes out. And justifying each step.

2.2 A variant

Exercise. Let A and B be two sets. Show that

$$A \cup B = A\backslash B \cup B\backslash A \cup A \cap B.$$

Figure 4: An exercise exactly equivalent to that of Fig. 3.

[2] and a well-known one.

3 Inductive proof

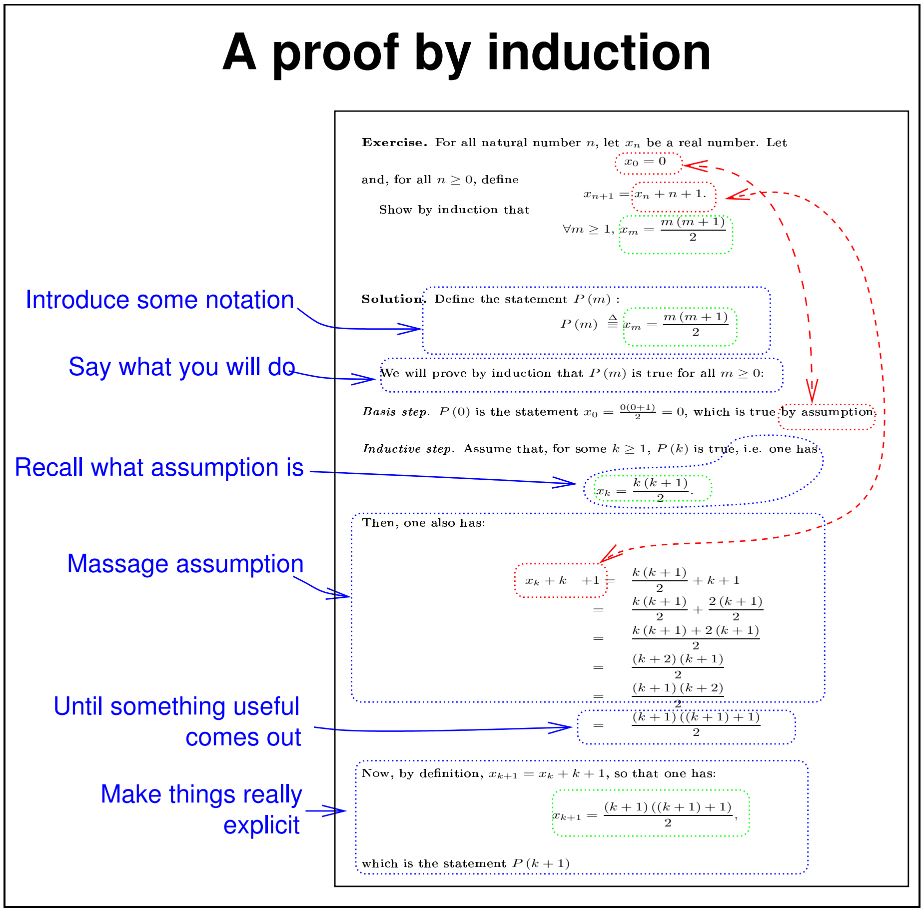

Figure 5: Inductive proof

A proof by induction serves to prove that a certain statement about natural numbers is true for all natural numbers.

Step 1: Say what you will do

So I start by saying what statement I will show to be true and give it a name - $P(n)$:

Define the statement $P(m)$:

$$P(m) \triangleq x_m = \frac{m(m+1)}{2}$$

and I then say what I will do: prove the $P(m)$ is true for all $m \geq 0$.

Step 2: Show the statement true for $k = 0$

Then, by writing “Basis step,” I announce that I will now show that $P(0)$ is true.

Step 3: Show that $P(k)$ implies $P(k+1)$

In the “Induction step,” which I also announce clearly, I show that $P(k)$ implies $P(k+1)$. So, as usual, in order to prove that “|||| implies \\\\”, I start by supposing “||||” and then chaining logical equivalences, deductions etc until the desired statement “\\\\” comes out.

In the present case I assume $P(k)$ and pretty quickly, the desired statement $P(k+1)$ comes out and I point this out clearly.