CS275 WEEK 12 RECITATION EXERCISES - SOLUTION

For each question, read **each word** with the greatest care and **without hurrying**. If you have doubts about what is asked, **go back** to the wording of the question until the meaning of the question is clear. Then try to find an answer. If you get stuck, **contact** your T.A. or me.

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1. Reminders from two weeks ago

1.1. Big-O, small-o.

This is not really part of the syllabus, but may be of use.

A function g defined on the natural numbers is said to **asymptotically bound** a function f if, when n is big, $f(n)$ is not greater than a constant times $g(n)$. One then writes

$$f(n) \in O(g(n)).$$

Using a more mathematical notation, this definition is : There exist positive constants C and N such that, if $n > N$, one has

$$f(n) < C\,g(n).$$

A function g is said to **asymptotically dominate** a function f if, when n is big, $\frac{f(n)}{g(n)}$ tends to zero. One then writes

$$f(n) \in o(g(n)).$$

This definition, in mathematical notation, says that $\frac{f(n)}{g(n)}$ can be made as small as one wants, when one increases n. More precisely, for any (small) $\varepsilon > 0$, and if n is big enough, one has $\frac{f(n)}{g(n)} < \varepsilon$. This is written

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n > N, \frac{f(n)}{g(n)} < \varepsilon.$$

Here, N is the point at which n is "big enough."

2. Exercises

Exercise 1. Solve Exercise 12 p. 555 of [1]. What does the degree of a vertex represent in the acquaintanceship graph, where vertices represent all the people in the world? What do isolated and pendant vertices in this graph represent? In one study it was estimated that the average degree of a vertex in this graph is 1000. What does this mean in terms of the model?

Solution:

- There´s an edge between 2 individuals iff they know each other, for some definition of "knowing each other."
- The degree of a vertex is thus the number of individuals (s)he knows.
- Isolated vertices correspond to people who do not know anyone. E.g. the last survivor of an isolated population. Depending on the definition of "knowing each other," some amnesic people may be considered as isolated vertices, as well as any person who lives entirely secluded.
- Pendant vertices correspond to people who know a sinlge other person. The last two survivors of an isolated population fit this definition. Again, depending on the chosen definition "knowing each other," other people can be added, such as confined prisonners who only see their one and only warden.

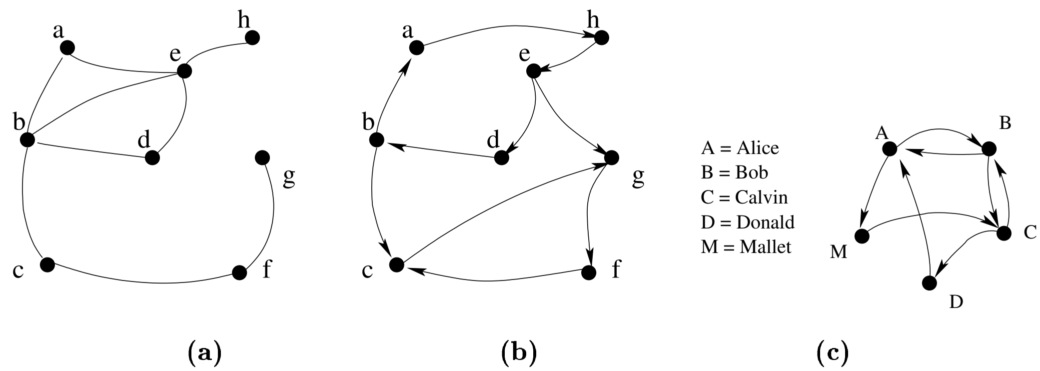


FIGURE 2.1. **(a)** Graph for Ex. 5. **(b)** Graph for Ex. 5. **(c)** Possible solution to Ex. 3.

- The most simple interpretation is that, "on average, an individual knows 1000 other individuals." Since the author does not cite the study[1], it appears hard to look it up and get a more precise interpretation.

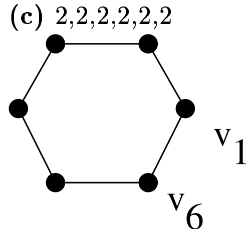
Exercise 2. Solve Exercise 28 p. 555 of [1]. Does there exist a simple graph with six vertices of these degrees? If so, draw such a graph.

Solution:

(a) 0,1,2,3,4,5
No: the sum of degrees is odd.

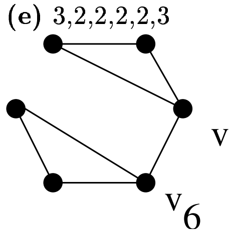
(b) 1,2,3,4,5,6
No: the sum of degrees is odd.

(c) 2,2,2,2,2,2

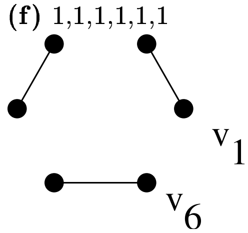


(d) 3,2,3,2,3,2
No: the sum of degrees is odd.

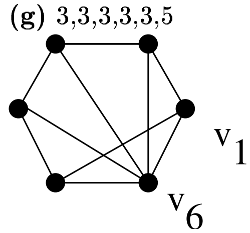
(e) 3,2,2,2,2,3



(f) 1,1,1,1,1,1



(g) 3,3,3,3,3,5



(h) 1,2,3,4,5,5
No: v_6 and v_5, having degree 5, are adjacent to all other vertices, and thus to v_1, so that $\deg(v_1) \geq 2$, which contradicts $\deg(v_1) = 1$.

Exercise 3. Draw the call graph as a directed graph, if

- Alice called Bob,
- Bob called Alice,
- Bob called Calvin,
- Calvin called Bob,
- Alice called Mallet,
- Donald called Alice,
- Calvin called Donald and
- Mallet called Calvin.

Write in mathematical notation the sets of vertices and edges of this graph.
Solution:

[1] Incidentally, not citing one´s source is very bad practice!

- See Fig. 2.1, (c).
- Writing A for "Alice", etc, the vertices are $V = \{A, B, C, D, M\}$.
- Edges : $E = \{(A, B), (A, M), (B, A), (B, C), (C, B), (C, D), (D, A), (M, C)\}$. Just make sure you write edges (A, B) and not $\{A, B\}$.

Exercise 4. Consider the simple graph of Figure 2.1, (a). Write down the following elements:

- Vertices, edges.
Solution: $V = \{a, \ldots, h\}$, $E = \{\{a, b\}, \{a, e\}, \{b, c\}, \{b, d\}, \{b, e\}, \{c, f\}, \{d, e\}, \{e, h\}, \{f, g\}\}$
- Degree of each vertex.
Solution: g, h have degree 1, a, c, d, f have degree 2 and b, e have degree 4.
- Isolated and pendant vertices.
Solution: No isolated vertices. g and h are pendant.
- Connected components of the graph.
Solution: There is a single connected component: V.

In the same figure, find, if possible

- A path starting at h, passing through a and ending at g.
Solution: (h, e, a, b, c, f, g).
- Two simple paths starting at h, passing through d and ending at g.
Solution: (h, e, d, b, c, f, g), $(h, e, d, b, e, a, b, c, f, g)$. This last path is a simple path, since each of the traversed edges $(\{e, h\}, \{d, e\}, \{b, d\}, \{b, e\}, \{a, e\}, \{a, b\}, \{b, c\}, \{c, f\}, \{f, g\})$ is traversed only once. Recall that a path is "simple" as long as it does not pass multiply through a given *edge*.
- A simple cycle starting at a and passing through g.
Solution: A cycle passing through g would necessarily have the subsequence (f, g, f), so that the edge $\{f, g\}$ would be found twice in the cycle, which thus cannot be simple.
- A cycle starting at h and passing through g.
Solution: $(h, e, b, c, f, g, f, c, b, e, h)$.

Exercise 5. Consider the directed graph of Figure 2.1, (b). What are:

- The vertices, edges.
Solution: $V = \{a, \ldots, h\}$, $E = \{(a, h), (b, a), (b, c), (c, g), (b, a), (d, b), (e, d), (f, c), (g, f), (h, e)\}$
- The in-degree, out-degree and degree of each vertex.
Solution: The in-degre, $\deg^-()$ of a, b, d, e, f, h is 1, that of c, g is 2. The out-degre, $\deg^+()$ of a, c, d, f, g, h is 1, that of b, e is two. The degree of a, d, f, h is 2, that of b, c, e, g is 3.
- Isolated and pendant vertices.
Solution: No isolated or pendant vertices.
- Weakly and strongly connected components of the graph.
Solution: $\{a, h, e, d, b\}$ and $\{c, g, f\}$ are the strongly connected components, and V is the only weakly connected component.

References

[1] K. H. Rosen. *Discrete Mathematics and Its Applications*. Mc Graw Hill, 5 edition, 2003.