DISCRETE MATH - NON-GRADED EXERCISES FOR 2005/09/13

For each question, read **each word** with the greatest care and **without precipitation**. If you have doubts about what is asked, **go back** to the wording of the question until the meaning of the question is clear. Then try to find an answer.

Some or all of these exercises will be solved during the Monday morning recitation. These exercises will not be graded.

Exercise 1.

a) Write in English the following statement about natural numbers:

$$\forall m\, \exists n,\ m - 8n < 7 \wedge m - 8n \geq 0. \tag{1.1}$$

b) Prove or disprove this statement.
Solution: Take $m = 7$. The statement $m - 8n \geq 0$ implies $n = 0$. If $n = 0$, then $m - 8 \cdot 0 = m \nleq 8$. So the statements $m - 8n < 7$ and $m - 8n \geq 0$ cannot be simultaneously satisfied and the whole proposition is false.

c) Write (in mathematical notation) the negation of proposition (1.1).
Solution: Each of the equivalent propositions below are negations of proposition (2.1). The last would be the "best".

	$\neg\,(\forall m\, \exists n,\ m - 8n < 7 \wedge m - 8n \geq 0)$	
$\equiv$	$\exists m\, \forall n,\ \neg\,(m - 8n < 7 \wedge m - 8n \geq 0)$	Negate quantifiers
$\equiv$	$\exists m\, \forall n,\ \neg\,(m - 8n < 7) \vee \neg\,(m - 8n \geq 0)$	Negate "and"
$\equiv$	$\exists m\, \forall n,\ (m - 8n \geq 7) \vee (m - 8n < 0)$	Negate parens
$\equiv$	$\exists m\, \forall n,\ m - 8n \geq 7 \vee m - 8n < 0$	Remove parens

Exercise 2.

a) Write in English the following statement about natural numbers:

$$\forall m\, \forall n\ (\exists p,\ n \leq p \wedge p | m) \Longrightarrow n \leq m. \tag{2.1}$$

Solution: For all natural numbers m, n, if there is a number p, that is greater than n, and that divides m, then n is smaller than or equal to m.

b) Prove or disprove this statement.
Solution: Assume the $(\exists p\, n \leq p \wedge p|m)$ is true. Then 1) $m = kp$ for some $k \geq 1$ because $p|m$. 2) $kp \geq kn$ because $p \geq n$. 3) $kn \geq n$ because $k \geq 1$. 4) Altogether $m \geq kp \geq kn \geq n$.

c) Write (in mathematical notation) the negation of proposition (2.1).
Solution: Each of the equivalent propositions below are negations of proposition (2.1). The last would be the "best".

	$\neg\,(\forall m\, \forall n\ (\exists p,\ n \leq p \wedge p|m) \Longrightarrow n \leq m)$	
$\equiv$	$\exists m\, \exists n\, \neg\,((\exists p,\ n \leq p \wedge p|m) \Longrightarrow n \leq m)$	Negate quantifiers
$\equiv$	$\exists m\, \exists n\ \ ((\exists p,\ n \leq p \wedge p|m) \wedge n > m)$	Use $\neg\,(A \Longrightarrow B) \equiv A \wedge \neg B$
$\equiv$	$\exists m\, \exists n\, \exists p,\ n \leq p \wedge p|m \wedge n > m$	Remove parentheses

Exercise 3. Let $P(x)$ denote a propositional function on the set of natural numbers. Write in mathematical notation, without using the notation $\exists!$, the proposition:

> There exists a unique x that verifies $P(x)$

Solution: This proposition is equivalent to (use "such that")

> There exists a unique x such that $P(x)$ is true

which is the same as ("is true" is not needed)

> There exists a unique x such that $P(x)$

which is the same as (express unicity)

> There exists a x such that $P(x)$ and it is the only object that verifies it

which is the same as (rephrase unicity)

> There exists a x such that $P(x)$ and no object y distinct from x verifies $P(y)$

which is the same as (write the condition on y in mathematical terms)

> There exists a x such that $P(x)$ and there does not exist a y such that $(y \neq x \wedge P(y))$

which is the same as (remove "such that" and "a")

> There exists x, $P(x)$ and there does not exist y, $(y \neq x \wedge P(y))$

which is the same as (write English in "Logic" style)

> There exists x, $P(x)$ and not (there exists y, $(y \neq x \wedge P(y))$)

which is the same as ("there exists", "and" and "not" become $\exists$, $\wedge$ and $\neg$, respectively)

> $\exists x,\, P(X) \wedge \neg(\exists y,\, y \neq x \wedge P(y))$

You can further simplify and obtain (negate the inner $\exists$)

> $\exists x,\, P(X) \wedge \forall y,\, \neg(y \neq x \wedge P(y))$

and then (negate $\wedge$)

> $\exists x,\, P(X) \wedge \forall y,\, y = x \vee \neg P(y)$

You may want to write the last statement as the equivalent implication (use definition of $\Longrightarrow$)

$$\exists x,\, P(X) \wedge \forall y,\, P(y) \Longrightarrow y = x$$

Note. You should recall from class that this is usually written with the shorthand notation

$$\exists! x\, P(x)$$

Exercise 4. Let $\mathbb{R}$ be the set of real numbers. Write in mathematical notation (possibly using $\exists!$) the sentence:

> There exists a unique real number x that verifies $ax + b = c$.

Determine whether this statement is true for all strictly positive natural numbers a, b and c. This exercise is somewhat like Ex. 54 p. 76 in [1].
Solution: By removing "that verifies" and word-to-word translation, one obtains:

$$\exists! x \;\; ax + b = c \tag{4.1}$$

Now for the second part. Suppose a, b and c are strictly positive natural numbers. Let´s examine the condition

$$ax + b = c. \tag{4.2}$$

This is equivalent to

$$ax = c - b$$

and, because $a \neq 0$, to

$$x = (c - b)/a \tag{4.3}$$

and indeed x is a real number.
So, for any a, b, $c \in \mathbb{N}^+$, there exists the real number $(c - b)/a$ and, calling this number x, it verifies Equation (4.2).
Moreover, in the steps from Eq. (4.2) to Eq. (4.3), we have seen that any number that verifies Eq. (4.2) also verifies Eq. (4.3). So one also has

$$\forall y,\, ax + b = c \Longrightarrow y = (c - b)/a.$$

Formally, one could say

$$\exists x,\, x \in \mathbb{R} \wedge x = \frac{c-b}{a} \wedge ax + b = c \wedge \forall y\, ay + b = c \Longrightarrow y = x.$$

Finally, one can remove the $x = \frac{c-b}{a}$ statement (use the tautology $A \wedge B \Longrightarrow A$) and use the shorthand notation $\exists!$ to obtain Eq. (4.1).

Exercise 5. We have a computer with infinite memory, that runs the following code quicker than you can read it.

```
infinitely long int i;
float [∞] x;
x[0] = 8.0;
for (i=0; i<=∞; i++)
     x[i+1] = 0.5*x[i] + rand()/(RAND_MAX+0.0);

// Note: rand() returns a value between 0 and RAND_MAX.
```

At this point, let´s see what can be said about the contents of the array `x` .

a) Write (possibly useful) upper and lower bounds on `x[1]`, `x[2]`, `x[3]` and `x[4]`.
Solution:
$4 \leq$ `x[1]` $\leq 4 + 1$,
$2 \leq$ `x[2]` $\leq 2 + \frac{1}{2} + 1$,
$1 \leq$ `x[3]` $\leq 1 + \frac{1}{4} + \frac{1}{2} + 1$,

$\frac{1}{2} \leq \texttt{x[4]} \leq \frac{1}{2} + \frac{1}{8} + \frac{1}{4} + \frac{1}{2} + 1$;
this suggests, but does not prove

$$\begin{aligned} 2^{3-n} \quad \leq \texttt{x[n]} \leq \quad & 2^{3-n} + \left(1 + ... + 2^{1-n}\right) \\ & = 2^{3-n} + \left(2 - 2^{1-n}\right) \\ & = 2 + 2^{1-n}\left(2^2 - 1\right) \\ & = 2 + 6 \cdot 2^{-n}. \end{aligned}$$

b) Let $Q(n)$ be the proposition $\texttt{x[}n\texttt{]} \geq 0$. Can you prove (e.g. by induction) that it is true for all n?

Solution: Do a proof by induction, i.e. show that $Q(0)$ holds and then that, if for some k, $Q(k)$ holds, then it holds for $Q(k+1)$:

1) $Q(0)$ is true because $\texttt{x[0]} = 4 \geq 0$.
2) Suppose $Q(k)$ holds, i.e. that $\texttt{x[}k\texttt{]} \geq 0$. Then, $\frac{1}{2}\texttt{x[}k\texttt{]} \geq 0$ and, since $\texttt{rand()} \geq 0$ always holds, one has $0 \leq \frac{1}{2}\texttt{x[}k\texttt{]} + \frac{\texttt{rand()}}{\texttt{RAND\_ MAX}} = \texttt{x[}k+1\texttt{]}$

c) What about $\texttt{x[}n\texttt{]} > 0$? (treacherous question[1])

Solution: If the $\texttt{x[}n\texttt{]}$ were reals, one would have $\forall n$, $\texttt{x[}n\texttt{]} \geq 2^{3-n} > 0$ (can be shown inductively).

But a float divided by 2 too many times becomes zero. So if `rand()` returns plenty of consecutive zeros, $\texttt{x[}n\texttt{]}$ will become zero, so one cannot rule out that $\exists n$, $x[n] = 0$.

Now[2], if `rand()` is properly implemented, this possibility becomes a *certainty* : in an infinite random sequence taking values in a finite set, all finite subsequences will appear at least once, w/ probability 1, including sequences of zeros of arbitrary length that will bring $\texttt{x[}n\texttt{]}$ to zero.

d) Let $P(n)$ be the proposition $\texttt{x[}n\texttt{]} < 3$. Can you show that it is true starting at a certain point? (That is, there exists an N such that $P(n)$ holds for all n greater than N; that is, $n \geq N \Longrightarrow P(n)$).

Solution:

1) $P(3)$ is true.
2) Suppose $P(k)$ holds for some $k \geq 3$.
   Then, $x[k+1] \leq \frac{1}{2}x[k] + 1 = \frac{3}{2} + 1 < 3$. Q.E.D.

e) What about the proposition $\texttt{x[}n\texttt{]} \leq 2$? (treacherous question)

Solution: Since it is easy to show that $\texttt{x[}n\texttt{]} \leq 2 \Longrightarrow \texttt{x[}n+1\texttt{]} \leq 2$, the harder question is "is there an n s.t. $\texttt{x[}n\texttt{]} \leq 2$?"

Again, there is a question of rounding and a question of probability[3]. Even if `rand()` returns only `RAND_MAX` values, the float $2 + 6 \cdot 2^{-n}$ will end up being rounded to 2. Even if the $\texttt{x[}n\texttt{]}$ were true real numbers in $\mathbb{R}$, `rand()` would not return only `RAND_MAX`. With probability one, for some $n_0 > 5$, `rand()/RAND_ MAX` would be smaller than $1/2$, so that $\texttt{x[}n_0+1\texttt{]} \leq 1 + 6 \cdot 2^{-n_0+1} + 1/2 \leq 1 + 6 \cdot 2^{-4} + 1/2 < 2$.

[1] Graded homeworks and exams will not contain "treacherous questions."
[2] This knowledge is given FYI. It is not part of CS275.
[3] These issues are outside the scope of CS275.

Exercise 6. Arithmetic sequence: Write the sum $0+4+8+\ldots+4n$ using the summation symbol $\sum$. What simple expression is this sum equal to?
Solution:

$$\sum_{i=0}^{n} 4i = 4\sum_{i=0}^{n} i = 4n\left(n+1\right)/2 = 2n\left(n+1\right).$$

Exercise 7. Arithmetic sequence: Write the sum $a+(a+3)+(a+6)+\ldots+(a+3n)$ using the summation symbol $\sum$. What simple expression is this sum equal to?
Solution:

$$\sum_{i=0}^{n} (a+3i) = a\sum_{i=0}^{n} 1 + 3\sum_{i=0}^{n} i = a\left(n+1\right) + 3n\left(n+1\right)/2.$$

Exercise 8. Geometric sequence: Write the sum $1+4+16+...+4^n$ using the summation symbol $\sum$. What simple expression is this sum equal to?
Solution:

$$\sum_{i=0}^{n} 4^i = \frac{4^{n+1}-1}{4-1} = \left(4^{n+1}-1\right)/3.$$

Exercise 9. Geometric sequence: Write the sum $b+4b+16b+...+4^n b$ using the summation symbol $\sum$. What simple expression is this sum equal to?
Solution:

$$\sum_{i=0}^{n} 4^i b = b\sum_{i=0}^{n} 4^i = b\frac{4^{n+1}-1}{4-1} = b\left(4^{n+1}-1\right)/3.$$

References

[1] K. H. Rosen. *Discrete Mathematics and Its Applications.* Mc Graw Hill, 5 edition, 2003.