1ST GRADED HOMEWORK

SOLUTIONS

Exercise 1. Let $A = \{0, 1, 2\}$. Write each of the following statements without using quantifiers, instead using just the $\wedge$, $\vee$ and $\neg$ operators.

a) $\exists x \in A,\ P(x)$
Solution: $P(0) \vee P(1) \vee P(2)$.

b) $\forall x \in A,\ P(x)$
Solution: $P(0) \wedge P(1) \wedge P(2)$.

c) $\exists x \in A,\ \neg P(x)$
Solution: $\neg P(0) \vee \neg P(1) \vee \neg P(2)$.

d) $\forall x \in A,\ \neg P(x)$
Solution: $\neg P(0) \wedge \neg P(1) \wedge \neg P(2)$.

e) $\neg \exists x \in A,\ P(x)$
Solutions: $\neg (P(0) \vee P(1) \vee P(2)) \equiv \neg P(0) \wedge \neg P(1) \wedge \neg P(2)$.

f) $\neg \forall x \in A,\ P(x)$
Solutions: $\neg (P(0) \wedge P(1) \wedge P(2)) \equiv \neg P(0) \vee \neg P(1) \vee \neg P(2)$.

Exercise 2. Write the definition of an odd number using mathematical notation.

Solution: A number $n \in \mathbb{Z}$ is odd if and only if $\exists m \in \mathbb{Z},\ n = 2m + 1$.
Note: This definition is composed of these elements:

[A number $n \in \mathbb{Z}$**]:** The object that has the property.
[is odd**]:** The name of the property (here, an adjective).
[if and only if**]:** The logical relation between "being odd" and the property at the right.
[$\exists m \in \mathbb{Z},\ n = 2m + 1$**]:** The meaning of "being odd" expressed in mathematical notation. In English, this means:

There exists an number m such that $n = 2m + 1$

or, equivalently,

$n = 2m + 1$ for some number m.

It is OK to consider odd natural numbers ($\mathbb{N}$) instead of integers ($\mathbb{Z}$).

Exercise 3. Show that the product of two odd numbers is odd (Rosen [1], p. 75, exercise 24).
Solution:

a) Let $m \in \mathbb{N}$ and $n \in \mathbb{N}$ be two odd numbers.
b) Being odd, there exist some m' and n' in $\mathbb{N}$ such that $m = 2m' + 1$ and $n = 2n' + 1$.
c) One has $mn = (2m' + 1)(2n' + 1) = 4m'n' + 2m' + 2n' + 1 = 2(2m'n' + m' + n') + 1$.
d) Because $mn = 2p + 1$ for $p = (2m'n' + m' + n')$, mn is odd.

Note: The question can also be stated as

Show that, if two numbers are odd, then their product is odd.

This is done (in the solution above) by, first, taking any two odd numbers and then showing that their product is indeed odd. The steps above do the following:

a) Take two odd numbers and name them m and n.
b) Explicitly write what it means that they are odd.
c) Write their product and chain equalities until it takes the "useful form" $mn = 2(...) + 1$.
d) Point out that by definition of being odd, the product is odd.

Exercise 4. Define the set of prime numbers using the "set builder notation," without using the word "prime".
Solution:
A number $n \in \mathbb{N}$ is prime if and only if $\forall m \in \mathbb{N},\, m|n \Longrightarrow m \in \{1, n\}$.
The set of prime numbers is thus

$$\{n \in \mathbb{N} \mid \forall m \in \mathbb{N},\, m|n \Longrightarrow m \in \{1, n\}\}.$$

Alternative solution:

$$\{n \in \mathbb{N} \mid \forall m \in \mathbb{N},\, \exists p \in \mathbb{N},\, n = mp \Longrightarrow m \in \{1, n\}\}.$$

Note: This definition of a set is composed of

[$\{\}$]: Brackets enclosing the whole definition. Read "the set of."
[$n \in \mathbb{N}$]: The name of the objects in the set, which will be natural numbers. Read "natural numbers n."
[$|$]: Separating vertical bar. Read "such that" or "that verify."
[$\forall m \in \mathbb{N},\, m|n \Longrightarrow m \in \{1, n\}$]: A definition of "$n$ is prime." This definition reads:

for all natural number m, if m divides n, then m is either 1 or n.

which is the same as:

the only divisors of n are 1 and n.

Exercise 5.

a)

1) Write in English the following statement about natural numbers:

$$\exists m \, \exists n \, \exists p, \; m^2 + n^2 = p^2.$$

Solution: "There exist natural numbers m, n and p such that the sum of the squares of m and n is the square of p."

2) Prove or disprove this statement.
Solution1: It is true, since $3^2 + 4^2 = 5^2$.
Solution2: Take $m = 3$, $n = 2$ and $p = 5$. Then $m^2 + n^2 = 3^2 + 4^2 = 25 = 5^2 = p^2$, so the statement is true.

b)

1) Write in English the following statement about natural numbers:

$$\forall m \, \forall n \, \forall p \; (m \mid n \wedge m \mid p) \Longrightarrow (\forall q \, \forall r, \; m \mid (qn + rp)) \, .$$

you may keep the mathematical notation $qn + rp$.
Solution: For all natural numbers m, n and p, if m divides both n and p, then for all natural numbers q and r, m divides $qn + rp$.

2) Prove or disprove this statement.
Solution: Take m, n and p such that $m|n$ and $m|p$.
Since $m \mid n$, one has $\forall q$, $m|qn$ (said in class).
Likewise, since $m \mid p$, one has $\forall r$, $m|rp$.
As said in class, $m|qn$ and $m|rp$ implies that $m| \, (qn + rp)$.
Note: This is a typical way of proving a "for all ... if /// then \\\" statement: it consists of the steps:
[Take m ...**]:** Recall the assumption "///".
[Since ... **]:** True statements that can be deduced from the assumption.
[implies that $m| \, (qn + rp)$**]:** The "\\\" statement that is thus deduced from the assumption.

Exercise 6. Let A and B be two non-empty sets. Prove or disprove that

$$A \times B = B \times A$$

if and only if

$$A = B.$$

Solution: In order to show that $A = B \iff A \times B = B \times A$, it is enough to show that:

a) $A = B \Longrightarrow A \times B = B \times A$ and
b) $A \times B = B \times A \Longrightarrow A = B$.

Point **a)** : suppose $A = B$. Then obviously $A \times B = B \times A$.
If this does not seem obvious, here is a more detailed proof: $A \times B = B \times A$, if and only if

$$\forall \, (x, y) \, ((x, y) \in A \times B \iff (x, y) \in B \times A) \, .$$

Suppose $(x, y) \in A \times B$. One has

$$
\begin{aligned}
(x,y)\in A\times B &\iff x\in A \wedge y\in B \\
&\iff x\in B \wedge y\in A \quad \text{Because } A=B \\
&\iff (x,y)\in B\times A.
\end{aligned}
$$

Point **b)** : In order to show that $A\times B=B\times A \implies A=B$, it is enough to show the equivalent statement:

$$A\neq B \implies A\times B\neq B\times A.$$

Suppose that $A\neq B$. Then, for some x,

$$x\in A\wedge x\notin B \vee x\in B\wedge x\notin A. \tag{6.1}$$

Since A and B are non-empty by hypothesis, there exist $a\in A$ and $b\in B$.
Note that $x\in A$ implies $(x,b)\in A\times B$ and that $x\notin B$ implies $(x,b)\notin B\times A$ (independently of whether $b\in A$ or not). Similarly, $(a,x)\in A\times B$ and $(a,x)\notin B\times A$. So Eq. (6.1) implies:

$$((x,b)\in A\times B)\wedge((x,b)\notin B\times A) \vee ((a,x)\in A\times B)\wedge((a,x)\notin B\times A),$$

which implies (by definition of the equality of sets)

$$A\times B\neq B\times A \vee A\times B\neq B\times A$$

and thus ($P\vee P\equiv P$ for any proposition P)

$$A\times B\neq B\times A.$$

Exercise 7. Write the negation of the following proposition without using the existential quantifier $\exists$.

$$\exists m\,\exists n\,(m>1 \wedge n>1 \wedge m^n-n^m=1).$$

Solution: Any one is correct:

$$
\begin{aligned}
&\forall m\,\forall n\,\neg(m>1 \wedge n>1 \wedge m^n-n^m=1) \\
\equiv\quad &\forall m\,\forall n\, m\leq 1 \vee n\leq 1 \vee m^n-n^m\neq 1
\end{aligned}.
$$

Exercise 8. Let variables range over $\mathbb{Z}$ and let $Q(x,y)$ be the propositional function $x+y=x-y$. Prove or disprove the following propositions (Rosen [1], p. 54, exercise 26):

Solution: Note that $x+y=x-y \iff x+y-x+y=0 \iff y=0$.

- **a)** $Q(1,1)$ is the statement $1=0$, which is false.
- **b)** $Q(2,0)$ is the statement $0=0$, which is true.
- **c)** $\forall y\, Q(1,y)$ is the statement $\forall y,\ y=0$, which is false.
- **d)** $\exists x\, Q(x,2)$ is the statement $\exists x,\ 2=0$, which is false.
- **e)** $\exists x\,\exists y\, Q(x,y)$ is the statement $\exists x\,\exists y,\ y=0$, which is true (take $x=y=0$).
- **f)** $\forall x\,\exists y\, Q(x,y)$ is the statement $\forall x\,\exists y,\ y=0$, which is true (take $y=0$ for any x).
- **g)** $\exists y\,\forall x\, Q(x,y)$ is the statement $\exists y\,\forall x,\ y=0$, which is true (take $y=0$).
- **h)** $\forall y\,\exists x\, Q(x,y)$ is the statement $\forall y\,\exists x,\ y=0$, which is false (take $y=1$).
- **i)** $\forall x\,\forall y\, Q(x,y)$ is the statement $\forall x\,\forall y,\ y=0$, which is false (take $x=y=1$).

Exercise 9. Show that, in propositional calculus:

a) $(P \Longrightarrow Q) \wedge (P \Longrightarrow R)$ and $P \Longrightarrow (Q \wedge R)$ are logically equivalent.
b) $\neg P \Longrightarrow (Q \Longrightarrow R)$ and $Q \Longrightarrow (P \vee R)$ are logically equivalent
c) $P \iff Q$ and $\neg P \iff \neg Q$ are logically equivalent.
d) $((P \vee Q) \wedge (\neg P \vee R)) \Longrightarrow (Q \vee R)$ is a tautology.

(this is Rosen [1], p. 27, exercises 20, 24, 26, 28). You may do so using truth tables or logical equivalence rules.

Solution:
Use e.g. truth tables. For this **make sure you know the truth tables** of the $\Longrightarrow$ and $\iff$ logical operators. Table 5, p. 6, Table 6, p. 9 and Table 3, p. 22. Alternatively, you may use the equivalences in Rosen [1] p. 24, do:

a) $(P \Longrightarrow Q) \wedge (P \Longrightarrow R) \equiv (\neg P \vee Q) \wedge (\neg P \vee R) \equiv \neg P \vee (Q \wedge R)$.
b) $\neg P \Longrightarrow (Q \Longrightarrow R) \equiv P \vee \neg Q \vee R \equiv Q \Longrightarrow (P \vee R)$.
c) $P \iff Q \equiv \neg P \iff \neg Q$ (straight out of table 7).
d) $((P \vee Q) \wedge (\neg P \vee R)) \equiv P \wedge (P \Longrightarrow R) \vee Q \wedge (P \Longrightarrow R)$ implies (modus ponens on left term) $R \vee Q \wedge (P \Longrightarrow R)$, which implies (because $Q \wedge (P \Longrightarrow R)$ implies Q) $R \vee Q$.

References

[1] K. H. Rosen. *Discrete Mathematics and Its Applications.* Mc Graw Hill, 5 edition, 2003.