CS275 RECITATION EXERCISES

For each question, read **each word** with the greatest care and **without hurrying**. If you have doubts about what is asked, **go back** to the wording of the question until the meaning of the question is clear. Then try to find an answer.

Be careful with notation and words:

$\{1,2,3\}$ is a **set** with elements 1, 2 and 3. The order does not matter, this is the same thing as $\{3,2,1\}$, $\{3,1,2\}$, $\{1,3,2\}$, $\{2,3,1\}$ and $\{2,1,3\}$.

When the elements in a set are all natural numbers, a set can also be called a **combination**. For example, the set $\{1,2,3\}$ is a combination of 3 out of the 9 elements of the set $\{1,...,9\}$.

$(1,2,3)$ is a **sequence** whose first element is 1, second is 2 and third is 3. The order does matter. This is not the same as $(2,1,3)$. This is sometimes called a **permutation with repetition**.

When the elements in a sequence are all distinct natural numbers, this sequence is also a **permutation**. $(3,2,1)$ is a permutation. $(2,1,2)$ is not.

Exercise 1. List all the permutations of two elements of the set $\{0,1,2\}$? List them in lexicographic order.
Solution: $(0,1)$, $(0,2)$, $(1,0)$, $(1,2)$, $(2,0)$, $(2,1)$

Exercise 2. List all the subsets of two elements of the set $\{0,1,2\}$? List them in lexicographic order.
Solution: $\{0,1\}$, $\{0,2\}$, $\{1,2\}$

Exercise 3. List all the sequences of two elements of the set $\{0,1,2\}$? List them in lexicographic order.
Solution: $(0,0)$, $(0,1)$, $(0,2)$, $(1,0)$, $(1,1)$, $(1,2)$, $(2,0)$, $(2,1)$, $(2,2)$

Exercise 4. List all the increasing sequences of two elements of the set $\{0,1,2\}$? List them in lexicographic order. Any resemblence with one of the cases above?
Solution: $(0,1)$, $(0,2)$, $(1,2)$

Exercise 5. For each of the statements below, find an object X that verifies it. Which of these statements are equivalent?

a) X is a permutation of p out of n elements[1].
Solution: $X = (1,...,p)$ is a permutation of p elements of $\{1,...,n\}$ (for any $p \leq n$).

b) X is a subset of p out of n elements.
Solution: $X = \{1,...,p\}$ is a subset of $\{1,...,n\}$ (for any $p \leq n$).

c) X is a permutation of p out of n elements with repetition.
Solution: $X = \underbrace{(1,...,1)}_{p}$ is a permutation w/ repetition of $\{1,...,n\}$ (for any $p \leq n$).

d) X is a sequence of p elements of the set $\{1...n\}$.
Solution: $X = \underbrace{(1,...,1)}_{p}$ is a sequence of p elements of the set $\{1,...,n\}$ (for any $p \leq n$).

e) X is a sequence of p distinct elements of the set $\{1...n\}$.
Solution: $X = (1,...,p)$ is a sequence of p distinct elements of $\{1,...,n\}$ (for any $p \leq n$).

f) X is a strictly increasing sequence of p elements of the set $\{1...n\}$.
Solution: $X = (1,...,p)$ is a strictly increasing sequence of p elements of $\{1,...,n\}$ (for any $p \leq n$).

[1] By default, permutation is "without repetition."

Solution: Statements **a** and **e** are equivalent. Statements **c** and **d** are equivalent. Statements **b** and **f** are not equivalent, although subsets p elements of $\{1, ..., n\}$ can be put in correspondence with strictly increasing sequences of p elements in $\{1, ..., n\}$. Note that, in statement **b**, X is a set (curly braces), while in statement **f**, it is a sequence (parens).

Exercise 6. What is the number of 5-element subsets of $\{0...7\}$?
Solution: $C(8, 5)$

Exercise 7. Let $(x_1, ..., x_5)$ be a sequence, where $x_i \in \{0...7\}$ for all i.

a) Write in mathematical notation the set of these sequences.
How many different sequences are there?
Solution:
Let $\mathcal{D} = \{0...7\}$. The set in question is

$$\{(x_1, ..., x_5) \mid \forall i,\, x_i \in \mathcal{D}\} = \underbrace{\mathcal{D} \times ... \times \mathcal{D}}_{5 \text{ times}} = \mathcal{D}^5.$$

by the product rule: $\left|\mathcal{D}^5\right| = |\mathcal{D}|^5 = 8^5 = 32768$.

b) Write in mathematical notation the set of sequences $(x_1, ..., x_5)$ where the x_i are two-by-two distinct.
How many such sequences are there?
Solution:
Let $\mathcal{D} = \{0...7\}$. The set in question is

$$E = \{(x_1, ..., x_5) \mid \forall i,\, x_i \in \mathcal{D},\, \forall j \neq i,\, x_i \neq x_j\}\,.$$

This is the set of permutations of 5out ouf 8elements. Show e.g. how to build this set.
$|E| = P(8, 5) = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 = \frac{8!}{(8-5)!} = 6720$.

c) Write in mathematical notation the set of sequences $(x_1, ..., x_5)$ where $x_1 < ... < x_5$.
How many such sequences are there?
Solution:
Let $\mathcal{D} = \{0...7\}$. The set in question is

$$F = \{(x_1, ..., x_5) \mid 0 \leq x_1 < \ldots < x_5 \leq 7\}\,.$$

Note that each 5-element subset of $\{0...7\}$ uniquely defines a strictly increasing sequence $(x_1, ..., x_5)$. Moreover, any strictly increasing sequence corresponds to a subset (set-to-sequence mapping is onto) and two distinct subsets define distinct sequences (set-to-sequence mapping is one-to-one). So the number of 5-element subsets of $\{0...7\}$ is equal to the number of elements in F $|F| = |\{A \mid A \subseteq \mathcal{D}, |A| = 5\}| = C(8, 5) = \frac{8!}{(8-5)!5!} = 56$

Exercise 8. Solve Exercise 14 p. 310 of the textbook [1]: how many distinct bit strings of length n, where n is a positive integer, start and end with 1s?
Solution: 2^{n-2}

Exercise 9. Solve Exercise 40 p. 312 of the textbook [1]: how many bit strings of length 7 either begin with two 0s or end with three 1s?
Hint: Inclusion-exclusion principle.
Solution: $2^{7-2} + 2^{7-3} - 2^{7-5} = 44$.

Exercise 10. Solve Exercise 14 p. 342 of the textbook [1]: how many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 = 17,$$

where x_1, x_2, x_3 and x_4 are nonnegative integers?

Hint: In Tuesday Sep. 21st class, x_i were number of fishes caught.
Solution:

$$C(17+4-1, 4-1) = C(20, 3) = 1140.$$

Exercise 11. Solve Exercise 6 p. 333 of the textbook [1]: what is the coefficient of x^7 in $(1+x)^{11}$?
Solution: $C(11, 7) = 330$.
Hint: Binomial theorem.

Exercise 12. Solve Exercise 8 p. 333 of the textbook [1]: what is the coefficient of x^8y^9 in the expansion of $(3x+2y)^{17}$?
Solution: $3^8 2^9 C(17, 8) \simeq 8.17E10$.

Exercise 13. Solve Exercise 16 p. 342 of the textbook [1]: how many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 29,$$

where x_i, for all $i \in \{1, 2, 3, 4, 5, 6\}$ is a nonnegative integer such that

- **a)** $x_i > 1$, for all $i \in \{1, 2, 3, 4, 5, 6\}$?
  Solution: $C(29+6-12, 6-1)$
- **b)** $x_i \geq i$, for all $i \in \{1, 2, 3, 4, 5, 6\}$?
  Solution:
- **c)** $x_1 \leq 5$?
  Solution:
- **d)** $x_1 < 8$ and $x_2 > 8$?
  Solution:

Exercise 14. Solve Exercise 10 p. 253 of the textbook [1]: prove that $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$ whenever n is a positive integer.
Solution:

References

[1] K. H. Rosen. *Discrete Mathematics and Its Applications*. Mc Graw Hill, 5 edition, 2003.