WEEK 7 : CS275 RECITATION EXERCISES

For each question, read **each word** with the greatest care and **without hurrying**. If you have doubts about what is asked, **go back** to the wording of the question until the meaning of the question is clear. Then try to find an answer.

Exercise 1. P. 480, n. 3 in [1], Section 7.1. For each of these relations on the set $\{1, 2, 3, 4\}$, decide whether it is reflexive, whether it is symmetric, whether it is antisymmetric, and whether it is transitive.

a) $\{(2,2), (2,3), (2,4), (3,2), (3,3), (3,4)\}$
Solution:
Not reflexive: $(1,1)$ is missing.
Not symmetric: $(2,4)$ and not $(4,2)$.
Not antisymmetric: $(2,3)$ and $(3,2)$ and $3 \neq 2$.
Transitive: $(2,3)$, $(3,4)$ and $(2,4)$. $(3,2)$, $(2,4)$ and $(3,4)$.

b) $\{(1,1), (1,2), (2,1), (2,2), (3,3), (4,4)\}$
Solution:
Reflexive: Just check.
Symmetric: Just check.
Not antisymmetric: $(1,2)$ and $(2,1)$ and $1 \neq 2$.
Transitive: Just check.

c) $\{(2,4), (4,2)\}$
Solution:
Not reflexive.
Symmetric: Just check.
Not antisymmetric: $(2,4)$ and $(4,2)$ and $4 \neq 2$.
Not transitive: $(2,4)$ and $(4,2)$ and not $(2,2)$.

d) $\{(1,2), (2,3), (3,4)\}$
Solution:
Not reflexive.
Not symmetric.
Antisymmetric.
Not transitive: $(1,2)$ and $(2,3)$ and not $(1,3)$.

e) $\{(1,1), (2,2), (3,3), (4,4)\}$
Solution:
Reflexive.
Symmetric.
Antisymmetric.
Transitive.

f) $\{(1,3), (1,4), (2,3), (2,4), (3,1), (3,4)\}$
Solution:
Not reflexive.
Not symmetric.
Not antisymmetric: $(1,3)$and $(3,1)$.
Not transitive: $(1,3)$, $(3,1)$ but not $(1,1)$.

Exercise 2. P. 480, n. 4 in [1], Section 7.1: Determine whether the relation R on the set of all people is reflexive, symmetric, antisymmetric, and/or transitive, where $(a, b) \in R$ iff

a) a is taller than b.
Solution:
Not reflexive: I am not taller than myself.
Not symmetric. X is taller than me, I am not taller than X.
Antisymmetric: Clearly, $\forall a\, \forall b\, \neg\left(R(a,b) \wedge R(b,a)\right)$ so that $\left(R(a,b) \wedge R(b,a)\right) \Longrightarrow a = b$ (recall that $F \Longrightarrow P$, for all statement P).
Transitive.

b) a and b were born on the same day.
Solution:
Reflexive.
Symmetric.
Not antisymmetric: Two distinct people may be born on the same day.
Transitive.

c) a has the same first name as b.
Solution:
Reflexive.
Symmetric.
Not antisymmetric: Two distinct people may have same 1st name.
Transitive.

d) a and b have a common grandparent.
Solution: Relation R would be like: $R(a,b) \equiv \text{GrandParents}(a) \cap \text{GrandParents}(b) \neq \emptyset$.
Reflexive.
Symmetric.
Not antisymmetric: Two distinct people may be have a common grandparent.
Not transitive: Let the grandparents of a be $\{u, v, w, x\}$, those of a´s cousin b be $\{u, v, y, z\}$ and those of b´s cousin c be $\{y, z, r, s\}$. Draw a family tree.

Exercise 3. P. 480, n. 5 in [1], Section 7.1: Determine whether the relation Ron the set of all Web pages is reflexive, symmetric, antisymmetric and/or transitive, where $(a, b) \in R$ iff

a) everyone who has visited Web page a has also visited Web page b.
Solution: R could be $R(a,b) \equiv \{x \,|\, x \text{ visited } a\} \subseteq \{x \,|\, x \text{ visited } b\} \equiv \forall x,\, x \text{ visited } a \Longrightarrow x \text{ visited } b$.
Reflexive.
Not symmetric.
Probably not antisymmetric (there must 2 pages out there that have been visited by exactly the same people).
Transitive.

b) there are no common links found on both Web page a and Web page b.
Solution: Relation R would be something like: $R(a,b) \equiv \text{LinksOutOf}(a) \cap \text{LinksOutOf}(b) = \emptyset$.
Not reflexive.
Symmetric.
Not antisymmetric: Two distinct pages may be have no common link.
Not transitive.

c) there is at least one common link found on both Web page a and Web page b.
Solution: Relation R would be something like: $R(a,b) \equiv \text{LinksOutOf}(a) \cap \text{LinksOutOf}(b) \neq \emptyset$.
Not reflexive: some pages have no outgoing links.
Symmetric.
Not antisymmetric: Two distinct pages may be have a common link.
Not transitive.

d) there is a Web page that includes links to both Web page a and Web page b.
Solution: Relation R would be something like: $R(a,b) \equiv \exists c,\, a \in \text{LinksOutOf}(c),\, b \in \text{LinksOutOf}(c)$.
Not reflexive, if one assumes there are pages w/ no link pointing to them. Else reflexive.

Symmetric.
Not antisymmetric.
Not transitive.

Exercise 4. P. 480, n. 6 in [1], Section 7.1: Determine whether the relation Ron the set of all real numbers is reflexive, symmetric, antisymmetric and/or transitive, where $(x, y) \in R$ iff

a) $x + y = 0$.
Solution:
Not reflexive $(1 + 1 \neq 0)$
symmetric $(x + y = 0 \Longrightarrow y + x = 0)$
not antisymmetric $(1 + -1 = 0 \wedge -1 + 1 = 0 \wedge -1 \neq 1)$
not transitive $1 + -1 = 0 \wedge -1 + 1 = 0 \wedge \neg(1 + 1 = 0)$.

b) $x = \pm y$.
Solution: Note $x = \pm y$ means $x \in \{-y, y\}$.
reflexive $(x = \pm x)$
symmetric $(x = \pm y \Longrightarrow y = \pm x)$
not antisymmetric $(1 = \pm(-1) \wedge (-1) = \pm 1 \wedge -1 \neq 1)$
transitive $x = \pm y \wedge y = \pm z \Longrightarrow x = \pm z$.

c) $x - y$ is a rational number.
Solution:
reflexive ($x - x = 0$ is rational)
symmetric ($x - y$ rational $\Longrightarrow y - x$ rational)
not antisymmetric ($1 - 2$ rational, $2 - 1$ rational and $2 \neq 1$)
transitive ($x - y$ rational and $y - z$ rational implies $x - z = x - y + y - z$ rational).

d) $x = 2y$.
Solution:
not reflexive $(1 \neq 2 \cdot 1)$
not symmetric ($6 = 2 \cdot 3$ and $3 \neq 2 \cdot 6$)
antisymmetric ($x = 2y$ and $y = 2x$ implies $x = 4x$ implies $x = 0$ and $y = 0$ and $x = y$)
not transitive ($24 = 2 \cdot 12$ and $12 = 2 \cdot 6$ but $24 \neq 2 \cdot 6$).

e) $xy \geq 0$.
Solution:
reflexive $(x^2 \geq 0)$
symmetric $(xy \geq 0 \Longrightarrow yx \geq 0)$
not antisymmetric ($1 \cdot 2 \geq 0$ and $2 \cdot 1 \geq 0$ and $1 \neq 2$)
not transitive ($-1 \cdot 0 \geq 0$ and $0 \cdot 1 \geq 0$ but $-1 \cdot 1 < 0$).

f) $xy = 0$.
Solution:
not reflexive $(1^2 \neq 0)$
symmetric $(xy = 0 \Longrightarrow yx = 0)$
not antisymmetric ($0 \cdot 2 = 0$ and $2 \cdot 0 = 0$ and $0 \neq 2$)
not transitive ($1 \cdot 0 = 0$ and $0 \cdot 2 = 0$ but $1 \cdot 2 \neq 0$).

g) $x = 1$.
Solution: Puzzled? Get on the ground by writing: $R(x, y) \equiv x = 1$ and $R(y, x) \equiv y = 1$
not reflexive $(\neg R(2, 2))$
not symmetric ($R(1, 2)$ and $\neg R(2, 1)$)
antisymmetric ($R(x, y)$ and $R(y, x)$ implies $x = 1$ and $y = 1$ and thus $x = y$)
transitive ($R(x, y)$ and $R(y, z)$ implies $x = 1$ and thus $R(x, z)$.

h) $x = 1$ or $y = 1$.
Solution: Note: $R(x, y) \equiv x = 1$ and $R(y, x) \equiv y = 1$
not reflexive $(\neg R(2, 2))$
symmetric $(R(x, y) \Longrightarrow R(y, x)$.
not antisymmetric ($R(1, 2)$ and $R(2, 1)$ and $2 \neq 1$)
not transitive ($R(3, 1)$ and $R(1, 2)$ and $\neg R(3, 2)$.

Exercise 5. P. 513, n. 1 in [1], Section 7.5: Which of these relations on the set $\{0, 1, 2, 3\}$ are equivalence relations? Determine the properties of an equivalence relation that the others lack.

a) $\{(0,0), (1,1), (2,2), (3,3)\}$ .... Equiv.
b) $\{(0,0), (0,2), (2,0), (2,2), (2,3), (3,2), (3,3)\}$
.... Not refl.:$(1,1)$ missing; not tran.: $(0,3)$ missing.
c) $\{(0,0), (1,1), (1,2), (2,1), (2,2), (3,3)\}$ .... Equiv.
d) $\{(0,0), (1,1), (1,3), (2,2), (2,3), (3,1), (3,2), (3,3)\}$ .... Not tran.: $(1,2)$ missing.
e) $\{(0,0), (0,1), (0,2), (1,0), (1,1), (1,2), (2,0), (2,2), (3,3)\}$
.... Not sym.: $(2,1)$ missing. Not tran.: $(2,1)$ missing.

Exercise 6. P. 514, n. 20 in [1], Section 7.5: What the equivalence classes of the equivalence relations in the previous exercise.

a) $\{(0,0), (1,1), (2,2), (3,3)\}$ .... $\{(0,0)\}, \{(1,1)\}, \{(2,2)\}, \{(3,3)\}$
c) $\{(0,0), (1,1), (1,2), (2,1), (2,2), (3,3)\}$ .... $\{(0,0)\}, \{(1,1), (2,2)\}, \{(3,3)\}$

Exercise 7. P. 513, n. 2 in [1], Section 7.5: For each of these relations on the "set" of all people are equivalence relations? Determine the properties of an equivalence relation that the others lack.

a) $\{(a, b) \mid a$ and b are the same age$\}$ .... Equiv.
b) $\{(a, b) \mid a$ and b have the same parents$\}$ .... Equiv.
c) $\{(a, b) \mid a$ and b share a common parent$\}$ .... Not tran.
d) $\{(a, b) \mid a$ and b have met$\}$ .... Not tran.
e) $\{(a, b) \mid a$ and b speak a common language$\}$ .... Not tran.

Exercise 8. P. 514, n. 21 in [1], Section 7.5: What the equivalence classes of the equivalence relations in the previous exercise.

Solution:

a) The equivalence class of person x is $\bar{x} = \{y \,|\, y$ has same age as $x\}$. For all $n \in \mathbb{N}$, there is an equiv class $\{y \mid y$ is n years old$\}$.
b) The equivalence class of person x is $\bar{x} = \{y \,|\, y$ has same parents as $x\} = \{y \,|\, y$ brother or sister of $x\}$. For all couple (m, d) with children, there is an equivalence class $\{y \,|\, y$ is a child of $(m, d)\}$.

Exercise 9. P. 512, n. 4 in [1], Section 7.5: Define three equivalence relations on the set of students in your discrete mathematics class different from the relations discussed in the text. Determine the equivalence classes of these equivalence relations.

Solution: For example: same section, same person, same size of shoes, same SSN ... .

Exercise 10. P. 527, n. 1 in [1], Section 7.6: Which of these are posets?

a) $(\mathbb{Z}, =)$ .... Yes, since reflexive, antisymmetric and transitive.
b) $(\mathbb{Z}, \neq)$ .... No, since not reflexive.
c) $(\mathbb{Z}, \geq)$ .... Yes, since reflexive, antisymmetric and transitive.
d) $(\mathbb{Z}, \nmid)$ .... No, since not reflexive.

Exercise 11. P. 528, n. 11 in [1], Section 7.6: Find the lexicographic ordering of these n-tuples:

a) $(1,1,2), (1,2,1)$ .... $(1,1,2) \preceq (1,2,1)$
b) $(0,1,2,3), (0,1,3,2)$ .... $(0,1,2,3) \preceq (0,1,3,2)$
c) $(1,0,1,0,1), (0,1,1,1,0)$ .... $(0,1,1,1,0) \preceq (1,0,1,0,1)$

References

[1] K. H. Rosen. *Discrete Mathematics and Its Applications*. Mc Graw Hill, 5 edition, 2003.