WEEK 8 : CS275 RECITATION EXERCISES

For each question, read **each word** with the greatest care and **without hurrying**. If you have doubts about what is asked, **go back** to the wording of the question until the meaning of the question is clear. Then try to find an answer.

Finish the exercises from week 7.

Exercise 1. P. 109, n. 25 in [1], Section 1.8. Suppose that g is a function from A to B and f is a function from B to C.

a) Show that if both f and g are one-to-one functions, then $f \circ g$ is also one-to-one.
Solution: It is sufficient to show that the images $(f \circ g)(x)$ and $(f \circ g)(x')$ of two distinct elements x, x' of A are distinct.
Let $x \neq x'$ be elements of A. Then $g(x) \neq g(x')$ because g is one-to-one, and $f(g(x)) \neq f(g(x'))$ because f is one-to-one and thus $f \circ g$ is one-to-one.

b) Show that if both f and g are onto functions, then $f \circ g$ is also onto.
Solution: It is sufficient to show that any element of C has a pre-image by $f \circ g$.
Let z be an element of C. Then, since f is onto, there exists a $y \in B$ s.t. $f(y) = z$; Furthermore, since g is onto, there exists a $x \in A$ s.t. $g(x) = y$. One thus has $f(g(x)) = f(y) = z$.

Exercise 2. P. 109, n. 28 in [1], Section 1.8. Find $f \circ g$ and $g \circ f$ where $f(x) = x^2 + 1$ and $g(x) = x + 2$ are functions from $\mathbb{R}$ to $\mathbb{R}$.
Solution:

$$\begin{aligned} (f \circ g)(x) &= f(x+2) = (x+2)^2 + 1 = x^2 + 4x + 5. \\ (g \circ f)(x) &= g\left(x^2+1\right) = x^2 + 1 + 2 = x^2 + 3. \end{aligned}$$

Exercise 3. P. 110, n. 34 in [1], Section 1.8. Let f be the function from $\mathbb{R}$ to $\mathbb{R}$ defined by $f(x) = x^2$. Find

a) $f^{-1}(\{1\})$. .... $\left\{y \in \mathbb{R} \mid y^2 = 1\right\} = \{-1, 1\}$.
b) $f^{-1}(\{x \mid 0 < x < 1\})$. .... $\left\{y \in \mathbb{R} \mid 1 > y^2 > 0\right\} = (-1, 0) \cup (0, 1) = (-1, 1) \setminus \{0\}$.
c) $f^{-1}(\{x \mid x > 4\})$. .... $\left\{y \in \mathbb{R} \mid y^2 > 4\right\} = (-\infty, -2) \cup (2, +\infty)$.

Notation: The open interval $\{x \mid 0 < x < 1\}$ may also be written $]0, 1[$ or $(0, 1)$.
Reminder: if $f : A \longrightarrow B$ and $X \subseteq B$, $f^{-1}(X) = \{x \in A \mid f(x) \in X\}$.

Exercise 4. P. 110, n. 36 in [1], Section 1.8. Let f be a function from A to B . Let S and T be subsets of A. Show that

a) $f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T)$.
Solution:

$$\begin{aligned} f^{-1}(S \cup T) &= \{x \in A \mid f(x) \in S \cup T\} \\ &= \{x \in A \mid f(x) \in S \vee f(x) \in T\} \\ &= \{x \in A \mid f(x) \in S\} \cup \{x \in A \mid f(x) \in T\} \\ &= f^{-1}(S) \cup f^{-1}(T) \end{aligned}$$

b) $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$.
Solution:

$$\begin{aligned} f^{-1}(S \cap T) &= \{x \in A \mid f(x) \in S \cap T\} \\ &= \{x \in A \mid f(x) \in S \wedge f(x) \in T\} \\ &= \{x \in A \mid f(x) \in S\} \wedge \{x \in A \mid f(x) \in T\} \\ &= f^{-1}(S) \cap f^{-1}(T) \end{aligned}$$

Exercise 5. P. 507, n. 20 in [1], Section 7.4: Let R be the relation that contains the pair (a, b) if a and b are cities such that there is a direct non-stop airline flight from a to b. When is (a, b) in

a) R^2? .... The set of pairs of cities that can be reached using exactly one connection.
b) R^3? .... The set of pairs of cities that can be reached using exactly two connection.
c) R^*? .... The set of pairs of cities that can be reached using any number of connections.

Exercise 6. P. 507, n. 29 in [1], Section 7.4: Find the smallest relation containing the relation $\{(1,2), (1,4), (3,3), (4,1)\}$ that is

a) reflexive and transitive. .... Add $(1,1), (2,2), (4,4), (4,2)$.
b) symmetric and transitive. .... Add $(2,1), (2,4), (4,2), (1,1), (2,2), (4,4)$.
c) reflexive, symmetric and transitive. .... Same as above.

Exercise 7. P. 528, n. 8 in [1], Section 7.6: Which of these pairs of elements are comparable in the poset $(\mathbb{Z}^+, \mid)$.

a) 5, 15. .... $5 \mid 15$.
b) 6, 9. .... $6 \nmid 9$ and $9 \nmid 6$.
c) 8, 16. .... $8 \mid 16$.
d) 7, 7. .... $7 \mid 7$.

Exercise 8. P. 528, n. 9 in [1], Section 7.6: Find two incomparable elements in these posets.

a) $(\wp(\{0,1,2\}), \subseteq)$, ... $\{1\}$ and $\{2\}$.
b) $(\{1,2,4,6,8\}, \mid)$. .... 6 and 8.

Exercise 9. P. 529, n. 27 in [1], Section 7.6: Answer these questions for the poset $(\{3,5,9,15,24,45\}, \mid)$.

a) Find the maximal elements. .... 24 and 45.
b) Find the minimal elements. .... 3 and 5.
c) Is there a greatest element? .... No: 24 and 45 are incomparable and maximal.
d) Is there a least element? .... No: 3 and 5 are incomparable and minimal.
e) Find all upper bounds of $\{3,5\}$. .... 15 and 45.
f) Find the least upper bound of $\{3,5\}$, if it exists. .... 15.
g) Find all lower bounds of $\{15,45\}$. .... 15, 3, 5.
h) Find the greatest lower bound of $\{15,45\}$, if it exists. .... 15.

Exercise 10. P. 529, n. 29 in [1], Section 7.6: Answer these questions for the poset
$(\{\{1\}, \{2\}, \{4\}, \{1,2\}, \{1,4\}, \{2,4\}, \{3,4\}, \{1,3,4\}, \{2,3,4\}\}, \subseteq)$.

a) Find the maximal elements. .... $\{1,2\}, \{1,3,4\}, \{2,3,4\}$.
b) Find the minimal elements. .... $\{1\}, \{2\}, \{4\}$.
c) Is there a greatest element? .... No.
d) Is there a least element? .... No.
e) Find all upper bounds of $\{\{2\}, \{4\}\}$. .... $\{2,4\}, \{2,3,4\}$.
f) Find the least upper bound of $\{\{2\}, \{4\}\}$, if it exists. .... $\{2,4\}$.
g) Find all lower bounds of $\{\{1,3,4\}, \{2,3,4\}\}$. .... $\{3,4\}, \{4\}$.
h) Find the greatest lower bound of $\{\{1,3,4\}, \{2,3,4\}\}$, if it exists. .... $\{3,4\}$.

References

[1] K. H. Rosen. *Discrete Mathematics and Its Applications*. Mc Graw Hill, 5 edition, 2003.