WEEK 9 : CS275 RECITATION EXERCISES

For each question, read **each word** with the greatest care and **without hurrying**. If you have doubts about what is asked, **go back** to the wording of the question until the meaning of the question is clear. Then try to find an answer.

Exercise 1. P. 319, n. 4 in [1], Section 4.2. A bowl contains ten red balls and ten blue balls. A woman selects balls at random without looking at them.

a) How many balls must she select to be sure of having at least three balls of the same color?
Solution: 5. Pigeonhole principle guarantees that 3 or more will have same color. Let x_1, x_2 be the number of red and blue balls. $x_1 + x_2 = 5 \Longrightarrow x_1 \geq 2.5 \vee x_2 \geq 2.5$.

b) How many balls must she select to be sure of having at least three blue balls?
Solution: 13. I can pick 10 red and 2 blue balls, w/out reaching 3 blue balls. With $x_i \in \{1 \ldots 10\}$, one has $x_1 + x_2 \geq 13 \Longrightarrow x_2 \geq 13 - x_1 \geq 3$.

Exercise 2. P. 319, n.6 in [1], Section 4.2.

a) Let d be a positive integer. Show that among any group of $d+1$ (not necessarily consecutive) integers, there are two with exactly the same remainder when they are divided by d.
Solution: Let x_i, $1 \leq i \leq d$ be the number of integers y in the group, s.t. $y \bmod d = i - 1$. If $x_1 + \ldots + x_d \geq d + 1$, then $\exists i,\ x_i \geq (d+1)/d > 1$.

Exercise 3. P. 319, n. 12 in [1], Section 4.2. How many ordered pairs of integers (a, b) are needed to guarantee that there are two ordered pairs (a_1, b_1) and (a_2, b_2) such that $a_1 \bmod 5 = a_2 \bmod 5$ and $b_1 \bmod 5 = b_2 \bmod 5$?[1]
Solution: 26. Note that $((a_1 - a_2) \bmod 5,\ (b_1 - b_2) \bmod 5)$ may take 25 possible values $m_1 = (0,0), \ldots, m_5 = (0,4), \ldots, m_{25} = (4,4)$. Let x_i be the number of pairs s.t. $((a_1 - a_2) \bmod 5,\ (b_1 - b_2) \bmod 5) = m_i$. Etc...

Exercise 4. P. 319, n. 14 in [1], Section 4.2.

a) Show that if seven [distinct] integers are selected from the first 10 positive integers, there must be at least two pairs of these integers with the sum 11.
Solution: Let $A_1 = \{1, 10\}$, $A_2 = \{2, 9\}$, $A_3 = \{3, 8\}$, $A_4 = \{4, 7\}$, $A_5 = \{5, 6\}$ and let x_i be the number of integers (out of the seven) that belong to A_i. Since $x_1 + \ldots + x_5 = 7 = 5 + 2$, either two of the x_i are equal to 2, or one of the x_i is equal to 3. In both cases, one can find two pairs of integers that sum up to 11.

b) Is the conclusion of part **a)** true if six integers are selected rather than seven?
Solution: No. Only one pair can be found.

Exercise 5. P. 320, n. 40 in [1], Section 4.2. There are 51 houses on a street. Each house has an address between 1000 and 1099, inclusive. Show that at least two houses have addresses that are consecutive integers.
Solution: By contradiction. Let $n_1, \ldots, n_{51}$ be the sorted house address numbers. If no two numbers are consecutive, then $n_{i+1} \geq n_i + 2$ for all $1 \leq i < 51$. Thus, $n_{51} \geq 1000 + 2 + \ldots + 2 = 1100$.

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[1] The modulo of a by b, also pronounced "a modulo b" is defined by:

$$a \bmod b = a - b \left\lfloor \frac{a}{b} \right\rfloor \in [0, b).$$

Exercise 6. P. 707, n. 2 in [1], Section 10.1. Find the values, if any, of the Boolean variable x that satisfies these equations.

a) $x \cdot 1 = 0$0.
b) $x + x = 0$0.
c) $x \cdot 1 = x$0, 1.
d) $x \cdot \bar{x} = 1$None.

Exercise 7. P. 708, n. 7 in [1], Section 10.1. What values of the Boolean variables x and y satisfy $xy = x + y$?
Solution: $(x, y) = (0, 0)$ and $(x, y) = (1, 1)$, as seen from the table

x	y	xy	$x+y$
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	1

.

Exercise 8. P. 708, n. 26 in [1], Section 10.1. Find the duals of these Boolean expressions.

a) $x + y$ .... $x \cdot y$
b) $\bar{x}\bar{y}$ .... $\bar{x} + \bar{y}$
c) $xyz + \bar{x}\bar{y}\bar{z}$ .... $(x + y + z) \cdot (\bar{x} + \bar{y} + \bar{z})$
d) $x\bar{z} + x \cdot 0 + \bar{x} \cdot 1$ .... $(x + \bar{z}) \cdot (x + 1) \cdot (\bar{x} + 0) = (x + \bar{z}) \cdot \bar{x} = \bar{z} \cdot \bar{x}$

Exercise 9. For each of the Boolean functions below,

a) rewrite it in a simpler way, if this is possible;
b) find its sum-of-products expansion, and
c) find its product of sums expansion.

Solution:

a) $F(x, y) = x + y\bar{x}$
  1) No simplification.
  2) $xy + x\bar{y} + \bar{x}y$
  3) $\bar{x} + \bar{y}$
b) $F(x, y, z) = \bar{x}y + \bar{y}z + \bar{z}x$
  1) No simplification, or perhaps $(\bar{x}+\bar{y}+\bar{z}) \cdot (x + y + z)$.
  2) $\bar{x}\bar{y}z + \bar{x}y\bar{z} + \bar{x}yz + x\bar{y}\bar{z} + x\bar{y}z + xy\bar{z}$
  3) $(\bar{x}+\bar{y}+\bar{z}) \cdot (x + y + z)$
c) $F(x, y, z) = \bar{x}(yz + \bar{y}\bar{z}) + x(y\bar{z} + \bar{y}z)$
  1) No simplification, or perhaps $\bar{x}\bar{y}\bar{z} + \bar{x}yz + xy\bar{z} + x\bar{y}z$.
  2) $\bar{x}\bar{y}\bar{z} + \bar{x}yz + xy\bar{z} + x\bar{y}z$.
  3) $(x + y + \bar{z}) \cdot (x + \bar{y} + z) \cdot (\bar{x} + y + z) \cdot (\bar{x} + \bar{y} + \bar{z})$.
d) $F(x, y, z) = \bar{x}(y + z) + \bar{y}(x + z) + \bar{z}(x + y)$
  1) $\overline{(\bar{x}\bar{y}\bar{z} + xyz)}$.
  2) $\bar{x}\bar{y}z + \bar{x}y\bar{z} + x\bar{y}\bar{z} + xy\bar{z} + x\bar{y}z + \bar{x}yz$.
  3) $(x + y + z) \cdot (\bar{x} + \bar{y} + \bar{z})$.

References

[1] K. H. Rosen. *Discrete Mathematics and Its Applications.* Mc Graw Hill, 5 edition, 2003.